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SOME RESULTS ON INVARIANT SETS FOR TRANSLATION PARAMETER FAMILY OF PROBABILITY MEASURES—I

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1. Preliminaries. Let μ be a given probability measure on $(\mathbf{X}, \mathbf{B})$, where $\mathbf{X}$ is some finite dimensional Euclidean space and $\mathbf{B}$ is the class of Borel sets on $\mathbf{X}$. For each $\theta \in \mathbf{X}$, let $\mu_\theta(A) = \mu(A-\theta)$, where $A \in \mathbf{B}$ and $A-\theta = \{x: x+\theta \in A\}$. A set '$A$' is called *$\mu$-invariant* if $\mu_\theta(A) = \mu(A)\ \ \forall\, \theta \in \mathbf{X}$. For example the null set $\varnothing$ and the whole space $\mathbf{X}$ are trivially μ-invariant. The class of all μ-invariant sets is denoted by $\mathbf{A}(\mu)$. A set 'A' is called "non-trivial" if $0 < \mu(A) < 1$. That "non-trivial" μ-invariant sets exist is seen by noting that if μ assigns probability $\frac{1}{2}$ each to $\{0\}$ and $\{1\}$, then A is μ-invariant if and only if $A' = A+1$, e.g., $A = \bigcup_{-\infty}^{\infty} (2n, 2n+1]$ is μ-invariant with $\mu(A) = \frac{1}{2}$. It is easily seen that $\mathbf{A}(\mu)$ is a monotone class, is closed for complementation and disjoint unions, and is not necessarily a σ-algebra. The probability measure μ is called weakly incomplete (weakly-complete) if it has (or does not have) non-trivial μ-invariant sets.

The results we present here have originated from a paper of Basu and Ghosh (1969) on μ-invariant sets. Our main object is to make a careful study of some of the conjectures contained in their paper. A brief account of the results contained in the paper is as follows:

(i) Let $\hat{\mu}(t) = \int \exp\{i(t, x)\}d\mu(x)$ denote the Fourier transform of μ. Basu and Ghosh show that if $S(\mu) = \{t: \hat{\mu}(t) = 0\}$ consists of finitely many elements, then μ is weakly complete. We shall show that if $S(\mu)$ is compact, or contained in a certain coset of a closed subgroup, then μ is weakly complete.

(ii) Now let μ be a probability measure on $(\mathbf{R}^1, \mathbf{B})$ and suppose that $S(\mu) = \{\pm 1, \pm 2, \cdots\}$. In this case Basu and Ghosh show that $\mathbf{A}(\mu)$ consists of all Borel sets of period 2π, i.e. a Borel set $A \in \mathbf{A}(\mu)$ iff $A+2\pi = A$ a.e. We strengthen this result by showing that the same assertion concerning $\mathbf{A}(\mu)$ is true if $S(\mu) = \{\pm 1, \pm 2, \cdots\} \cup K \cup J$, where K is compact and J is contained in a certain coset of a closed subgroup. We thus prove a conjecture mentioned in Basu and Ghosh ((1969) Theorem 8, page 168). Basu and Blum, in a personal communication, have noted that, in this case, μ must necessarily be absolutely continuous.

(iii) Let $A \in \mathbf{A}(\mu)$. Then $\forall\, \theta \in \mathbf{X}$, $A-\theta \in \mathbf{A}(\mu)$. Thus $\mathbf{A}(\mu)$ is translation invariant. If $\mathbf{A}(\mu)$ is also a σ-algebra, then $\mathbf{A}(\mu)$ becomes a translation invariant σ-algebra. Now let H be a closed subgroup of $\mathbf{X}$ and $\mathbf{B}_H$ be the class of Borel sets E with the property that $E+h = E$ for every $h \in H$. Clearly $\mathbf{B}_H$ is translation invariant. We show that every translation invariant σ-algebra is of the $\mathbf{B}_H$ kind, i.e. the σ-algebra is the σ-algebra of Borel sets that are invariant for some closed subgroup H. An immedi-

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