AN EXTREMAL PROBLEM FOR QUASICONFORMAL MAPPINGS AND A THEOREM BY THURSTON

BY

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To Lars V. Ahlfors, on his 70th birthday

§ 1. Statement of the problem

The new extremal problem considered in this paper, and the form of the solution are suggested by Thurston's beautiful theorem on the structure of topological self-mappings of a surface (Theorem 4 in [21]). In treating this problem, however, we use none of Thurston's results and thus obtain also a new proof of his theorem.

Unless otherwise stated all surfaces considered in this paper will be assumed to be *oriented* and of finite type (p, m), that is homeomorphic to a sphere with $p \geqslant 0$ handles from which one has removed $m \geqslant 0$ disjoint continua. To avoid uninteresting special cases we assume that

$$2p-2+m>0. \tag{1.1}$$

All mappings between surfaces (or between finite disjoint unions of surfaces) will be assumed bijective, topological and orientation preserving. We recall (see Mangler [15]) that two mappings of a surface are isotopic if and only if they are homotopic.

A *conformal structure* on a surface S is a mapping σ of S onto a Riemann surface. If $f: S_1 \to S_2$ is a mapping, and σ_1, σ_2 are conformal structures on S_1 and S_2, respectively, then the deviation of $\sigma_2 \circ f \circ \sigma_1^{-1}$ from conformality is measured by the *dilatation*

$$K = K(\sigma_2 \circ f \circ \sigma_1^{-1}) = K_{\sigma_1, \sigma_2}(f).$$

We recall that $1 \leqslant K \leqslant +\infty$, with $K=1$ signifying that $\sigma_2 \circ f \circ \sigma_1^{-1}$ is conformal, and $K=+\infty$ signifying that this mapping is not even quasiconformal. If $S_1 = S_2$ and $\sigma_1 = \sigma_2$, we write $K_\sigma(f)$ instead of $K_{\sigma,\sigma}(f)$.

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