SIMPLICES OF MAXIMAL VOLUME IN HYPERBOLIC n-SPACE

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1. Introduction

Consider hyperbolic n-space H^n represented as the Poincaré disk model

$$H^n \sim D^n = \{\mathbf{x} \in \mathbf{R}^n \mid \|\mathbf{x}\| < 1\}$$

with the Riemannian metric

$$ds^2 = \frac{4}{(1-r^2)^2} \sum_{i=1}^{n} (dx_i)^2 \quad \text{where} \quad r^2 = \sum_{i=1}^{n} x_i^2.$$

The geodesics in H^n are the circles orthogonal to the "sphere at infinity"

$$\partial H^n = \{\mathbf{x} \in \mathbf{R}^n \mid \|\mathbf{x}\| = 1\} = S^{n-1}.$$

An *n-simplex* in H^n with vertices $\mathbf{v}_0, \ldots, \mathbf{v}_n \in H^n \cup \partial H^n$ is the closed subset of H^n bounded by the $n+1$ spheres which contain all the vertices except one and which are orthogonal to S^{n-1}. A simplex is called *ideal* if all the vertices are on the sphere at infinity. It is easy to see that the volume of a hyperbolic n-simplex is finite also if some of the vertices are on the sphere at infinity. A simplex is called *regular* if any permutation of its vertices can be induced by an isometry of H^n. This makes sense also for ideal simplices since any isometry of H^n can be extended continuously to $H^n \cup \partial H^n$. There is, up to isometry, only one ideal regular n-simplex in H^n.

The main result of the present paper is the following theorem which was conjectured by Thurston ([6], section 6.1).

THEOREM 1. *In hyperbolic n-space, for $n \geqslant 2$, a simplex is of maximal volume if and only if it is ideal and regular.*

Since any hyperbolic n-simplex is contained in an ideal one it suffices, when proving Theorem 1, to consider ideal simplices. We shall use the notation $\tau[n]$ for an arbitrary ideal n-simplex in H^n, while $\tau_0[n]$ always denotes a regular $\tau[n]$.