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# Wave breaking for nonlinear nonlocal shallow water equations

by

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## 1. Introduction

A basic question in the theory of nonlinear partial differential equations is: when can a singularity form and what is its nature? The typical well-posedness result (see e.g. [23]) asserts that either a solution of a PDE exists for all time or else there is a time $T<\infty$ such that some norm of the solution becomes unbounded as $t\uparrow T$; the latter phenomena is called (finite-time) blow-up.

The behavior of the solution as the blow-up time is approached is of particular interest. A simple kind of singularity occurs when the solution itself becomes unbounded in a finite time. For models describing water waves we say that wave breaking holds if the solution (representing the wave) remains bounded but its slope becomes infinite in finite time: the profile will gradually steepen as it propagates until it finally develops a point where the slope is vertical and the wave is said to have broken, cf. [35].

Blow-up techniques are quite particular to each type of equation; there is no general method [34]. We present now a quite representative sample of methods to accomplish blow-up for nonlinear wave equations; see also the recent surveys [2], [3], [34].

The functional method (see [17], [22]) consists in introducing an appropriate functional $F$ of a solution, depending on time, and using the PDE to get a (first- or second-order) differential inequality for $F$ which implies finite-time blow-up for well-chosen initial data; surveys can be found in [21], [34].

A more sophisticated method than the functional method is the averaging method: introducing appropriate coordinates it is sometimes possible to prove the breakdown of certain averaged quantities (see [2], [33]).

For semilinear equations it is possible to define the maximal domain of existence of a solution and to try to understand the behavior of the solution near the boundary of this domain; for the equation $u_{tt}-u_{xx}-|u|^p=0$ with $1<p<\infty$, any solution which blows up