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On the convergence of the product of independent random variables

By

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1. Introduction

Let $\{X_k\}$ be a sequence of integrable random variables on a probability space $(\Omega, \mathscr{B}, \mathrm{P})$, $\mathscr{B}_n$ be the σ-algebra generated by $\{X_k;\ 1 \leq k \leq n\}$, denote the mathematical expectation by E[ ] and the mathematical expectation on a set $A \in \mathscr{B}$ by E[; A].

$\{X_k\}$ is *upper semi-bounded* iff there exists a positive constant K such that

$$\sum_k \mathrm{E}[X_k;\ X_k \geq K] < +\infty .$$

If there exists a positive constant K such that $X_k < K$, a.s., $k \in \boldsymbol{N}$, then $\{X_k\}$ is upper semi-bounded.

Assume that $\{X_k;\ k \in \boldsymbol{N}\}$ are independent and upper semi-bounded with non-negative means. Then in Paragraph 2 we shall show the equivalence of the $\boldsymbol{L}^1$-convergence and the almost sure convergence of $\sum_k X_k$ (Theorem 1). Furthermore, assume that $X_k > -1$, a.s., and $\mathrm{E}[X_k] = 0$, $k \in \boldsymbol{N}$. Then in Paragraph 3 we shall show the equivalence of the almost sure convergence of $\sum_k X_k$ and the $\boldsymbol{L}^1$-convergence of $\prod_k (1+X_k)$ (Theorem 2). Note that if $\{x_k\}$ is a real sequence, then the convergence of $\sum_k x_k$ does not imply the convergence of $\prod_k (1+x_k)$ (for example $x_k = (-1)^k k^{-\frac{1}{2}}$). Conversely the convergence of $\prod_k (1+x_k)$ does not imply the convergence of $\prod_k (1+x_k)$ (for example $x_k = (-1)^k k^{-\frac{1}{2}} + (2k)^{-1}$). As an application in Paragraph 4 we shall give necessary and sufficient conditions for the equivalence (mutual absolute continuity) of two infinite product measures based on the convergence of marginal densities (Theorem 3).

2. Sum of semi-bounded independent random variables

In this paragraph we prove the following theorem.

Theorem 1. *Let* $\{X_k\}$ *be a sequence of upper semi-bounded independent random variables such that* $\mathrm{E}[X_k] \geq 0$, $k \in \boldsymbol{N}$. *Then all of the following statements are equivalent.*

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(A) $\sum_k X_k$ *converges in* $\boldsymbol{L}^1$.

(B) $\sup_n \mathrm{E}[|\sum_{k=1}^n X_k|] < +\infty$.

(C) $\sum_k X_k$ *converges almost surely.*

(D) $\sum_k X_k$ *and* $\sum_k X_k^2$ *converge almost surely.*

Proof. (A)⇒(B) and (D)⇒(C) are trivial. (B)⇒(C) is proved by the Doob's theorem since $S_n = \sum_{k=1}^n X_k$ is a $\mathscr{B}_n$-matringale (W. Stout [3], Theorem 2-7-2).

(C)⇒(D). Since $\{X_k\}$ is upper semi-bounded, there exists a positive constant K such that

$$\sum_k \mathrm{E}[X_k;\ X_k \geq K] < +\infty. \tag{1}$$

Define

$$Y_k = \begin{cases} X_k, & \text{if } \ |X_k| < K, \\ 0, & \text{otherwise}, \end{cases}$$

and $Z_k = X_k - Y_k$, $k \in N$. Then, since $\sum_k X_k$ converges almost surely, by Kolmogorov's three series theorem the following three series are convergent.

$$\sum_k \mathrm{P}(|X_k| \geq K) < +\infty, \tag{2}$$

$$\sum_k \mathrm{E}[Y_k] \quad \text{converges}, \tag{3}$$

$$\sum_k \{\mathrm{E}[Y_k^2] - \mathrm{E}[Y_k]^2\} < +\infty. \tag{4}$$

For every k in N define $m_k^+ = \mathrm{E}[X_k;\ X_k \geq K] \geq 0$, $m_k^0 = \mathrm{E}[Y_k] = \mathrm{E}[X_k;\ |X_k| < K]$, and $m_k^- = -\mathrm{E}[X_k;\ X_k \leq -K] \geq 0$. Then by the assumption we have

$$m_k^+ + m_k^0 - m_k^- = \mathrm{E}[X_k] \geq 0, \quad k \in \boldsymbol{N},$$

and by (1) and (3)

$$\sum_k m_k^- \leq \sum_k m_k^+ + \sum_k m_k^0$$

converges. Furthermore we have for every k in $\boldsymbol{N}$

$$m_k^+ + m_k^0 \geq m_k^0 \geq m_k^- - m_k^+ \geq -(m_k^+ + m_k^-)$$

so that

$$\sum_k |m_k^0| \leq \sum_k (m_k^+ + m_k^-) + \sum_k (m_k^+ + m_+^0) < +\infty.$$

Consequently $\sum_k m_k^0$ converges absolutely. This implies the convergence of $\sum_k \mathrm{E}[Y_k]^2$ and by (4) we have

$$\sum_k \mathrm{E}[Y_k^2] < +\infty. \tag{5}$$

By Kolmogorov's three series theorem (2) and (5) imply (D).

(C)⇒(A). For every m, $n(n<m)\in \boldsymbol{N}$ we have

$$\begin{aligned}\mathrm{E}[|\sum_{n<k\leq m} X_k|] &\leq \mathrm{E}[|\sum_{n<k\leq m} Y_k|] + \mathrm{E}[|\sum_{n<k\leq m} Z_k|] \\ &\leq \mathrm{E}[|\sum_{n<k\leq} Y_k|^2]^{\frac{1}{2}} + \sum_{n<k\leq m} \mathrm{E}[|Z_k|] \\ &\leq \{\sum_{n<k\leq m} \mathrm{E}[Y_k^2] + [\sum_{n<k\leq m} (m_k^- + m_k^+)]^2\}^{\frac{1}{2}} + \sum_{n<k\leq m} (m_k^+ + m_k^-)\end{aligned}$$

$\to 0$ as n, $m\to +\infty$. Therefore $\sum_k X_k$ converges in $\boldsymbol{L}^1$.

3. Infinite product of independentrandom variables

In this paragraph we extend Theorem 1 to the convergence of infinite product of independent random variables.

Theorem 2. *Let $\{X_k\}$ be a sequence of independent random variables such that $\mathrm{E}[X_k]=0$ and $X_k>-1$, a.s., $k\in \boldsymbol{N}$. Then all of the following statements are equivalent.*

(A) $\sum_k X_k$ *converges in* $\boldsymbol{L}^1$.

(B) $\sup_n \mathrm{E}[|\sum_{k=1}^n X_k|]<+\infty$.

(C) $\sum_k X_k$ *converges almost surely.*

(D) $\sum_k X_k$ *and* $\sum_k X_k^2$ *converge almost surely.*

(E) $\prod_k (1+X_k)$ *converges and is positive almost surely.*

(F) $\prod_k (1+X_k)$ *converges in* $\boldsymbol{L}^1$.

Proof. Since $\{-X_k\}$ is upper semi-bounded with zero mean, the equivalences from (A) to (D) are already proved in Theorem 1. (D)⇒(E) is proved by Lemma 8 of *H*. Sato [2].

(E)⇒(F). Since we have

$$\begin{aligned}\prod_{k=1}^{\infty} \mathrm{E}[\sqrt{(1+X_k)}] &= \liminf_n \mathrm{E}[\sqrt{\prod_{k=1}^{n} (1+X_k)}] \\ &\geq \mathrm{E}[\liminf_n \sqrt{\prod_{k=1}^{n} (1+X_k)}] = E[\sqrt{\prod_{k=1}^{\infty} (1+X_k)}]>0,\end{aligned}$$

the arguments of J. Neveu [1], Proposition III-1-2 imply (F).

(F)⇒(C). Assume that $V_n=\prod_{k=1}^{n} (1+X_k)$ converges in L^1. Then, since $\{V_n\}$ is a $\mathscr{B}_n$-martingale, V_n converges almost surely to $V=\prod_{k=1}^{+\infty} (1+X_k)$ and we have

$$\mathrm{E}[V]=\lim_n \mathrm{E}[V_n]=1,$$

so that $\mathrm{P}(V>0)>0$. Since $\{\log(1+X_k)\}$ is an independent random sequence, by the 0–1 law we have

$$P(V>0)=P(\sum_k \log(1+X_k) \text{ converges})$$
$$=0 \quad \text{or} \quad 1.$$

Therefore we have $V>0$, a.s..

On the other hand define

$$U_1=1,$$
$$U_k=X_k V_{k-1}, \quad k=2, 3, 4,\dots.$$

Then $\{U_k\}$ is a $\mathscr{B}_k$-martingale difference sequence such that

$$\sup_n E[|\sum_{k=1}^n U_k|]=\sup_n E[V_n]=1<+\infty.$$

Define

$$v_1=1,$$
$$v_k=V_{k-1}^{-1}, \quad k=2, 3, 4,\dots.$$

Then for every k in N, v_k is $\mathscr{B}_{k-1}$-measurable and we have

$$\sup_n |v_n|\leq \sup_n \prod_{k=1}^n (1+X_k)_-^{\,1}\leq \frac{1}{\inf_n \prod_{k=1}^n (1+X_k)}<+\infty, \quad \text{a.s.}.$$

Therefore by Burkholder's theorem (W. Stout [3], Theorem 2–9–4) $\sum_k X_k=\sum_k v_k U_k$ converges almost surely.

4. Absolute continuity of the infinite product measures

In this paragraph we apply Theorem 2 to the equivalence of two infinite product measures on the sequence space.

Theorem 3. *Let $\mu=\prod_k \mu_k$ and $\nu=\prod_k \nu_k$ be infinite product measures on the sequence space $\boldsymbol{R}^{\boldsymbol{N}}$, where $\{\mu_k;\, k\in \boldsymbol{N}\}$ and $\{\nu_k;\, k\in \boldsymbol{N}\}$ are probabilities on $\boldsymbol{R}^1$ such that $\nu_k\sim\mu_k$ (equivalent) for every k in $\boldsymbol{N}$. Then all of the following statements are equivalent.*

(A) $\sum_k \left(\frac{d\nu_k}{d\mu_k}(x_k)-1\right)$ *converges in $\boldsymbol{L}^1(\mu)$.*

(B) $\sup_n \int \left|\sum_{k=1}^n \left(\frac{d\nu_k}{d\mu_k}(x_k)-1\right)\right| d(\mu_1\times\mu_2\times\cdots\times\mu_n)<+\infty.$

(C) $\sum_k \left(\frac{d\nu_k}{d\mu_k}(x_k)-1\right)$ *converges almost surely (μ).*

(D) $\sum_k \left(\frac{d\nu_k}{d\mu_k}(x_k)-1\right)$ *and* $\sum_k \left(\frac{d\nu_k}{d\mu_k}(x_k)-1\right)^2$ *converges almost surely (μ).*

(E) $\prod_k \frac{d\nu_k}{d\mu_k}(x_k)$ *converges and is positive almost surely (μ).*

(F) $\nu \sim \mu$.

In the above statements $x_k = x_k(x)$, $k \in \boldsymbol{N}$, *denotes the k-th coordinate of* $x = \{x_k\} \in \boldsymbol{R}^{\boldsymbol{N}}$.

Proof. Define

$$X_k(x) = \frac{d\nu_k}{d\mu_k}(x_k) - 1, \quad x = \{x_k\} \in \boldsymbol{R}^{\boldsymbol{N}}, \quad k \in \boldsymbol{N}.$$

Then obviously the random sequence $\{X_k\}$ on the probability space $(\boldsymbol{R}^{\boldsymbol{N}}, \mu)$ satisfies te hypothesis of Theorem 2. Since the L^1-convergence of $\prod_k \frac{d\nu_k}{d\mu_k}(x_k) = \prod_k (1 + X_k)$ is equivalent to $\nu \sim \mu$ (J. Neveu [1], Proposition III–1–2), Theorem 3 is a special case of Theorem 2.

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References

[1] J. Neveu, Martingales à temps discrét, Paris, Masson & Cie, 1972.
[2] H. Sato, Characteristic functional of a probability measure absolutely continuous with respect to a Gaussian Radon measure, J. Func. Anal., **61** (1985), 222–245.
[3] W. Stout, Almost sure convergence, New York, San Fransisco and London, Academic Press, 1974.