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EXPONENTIAL STABILITY IN FUNCTIONAL DYNAMIC EQUATIONS ON TIME SCALES

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Abstract

We are interested in the exponential stability of the zero solution of a functional dynamic equation on a time scale, a nonempty closed subset of real numbers. The approach is based on suitable Lyapunov functionals and certain inequalities. We apply our results to obtain exponential stability in Volterra integrodynamic equations on time scales.

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1 Introduction

In this paper, we consider the exponential stability of the zero solution of functional equations of the form

$$x^{\Delta}(t) = G(t, x(s); 0 \le s \le t) := G(t, x(.)) \tag{1.1}$$

on a time scale $\mathbb{T}$ with $0 \in \mathbb{T}$, a closed subset of real numbers. x is a delta-differentiable $n \times 1$ vector function and $G : [0,\infty) \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is a given nonlinear continuous function in t and x such that $G(t,0) = 0$. "$.^{\Delta}$" turns out to be the usual derivative when $\mathbb{T} = \mathbb{R}$ and the forward difference operator when $\mathbb{T} = \mathbb{Z}$. The forward difference operator is defined by $\Delta x(t) = x(t+1) - x(t)$. Throughout this paper, for each continuous function $\phi : [0, t_0] \mapsto \mathbb{R}^n$ there exists at least one continuous function $x(t) = x(t, t_0, \phi)$ on an interval $[t_0, I]$ such that it satisfies (1.1) for $0 \le t_0 \le t \le I$ and such that $x(t, t_0, \phi) = \phi(t)$ for $0 \le t_0 \le I$. For the existence and extendibility of solutions of (1.1), we refer the reader to [4].

Our results in this paper extends the results in [8] to functional dynamic equations on time scales. Peterson and Raffoul in [8] obtain the exponential stability of the zero solution of the initial value problem

$$x^{\Delta}(t) = G(t, x(t)), \qquad x(t_0) = x_0, \qquad t_0 \ge 0, \quad x_0 \in \mathbb{R}$$

by using non-negative definite Lyapunov functions but their results do not apply to equations similar to the Volterra integrodynamic equations

$$x^{\Delta} = a(t)x + \int_0^t B(t,s) f(x(s)) \Delta s. \tag{1.2}$$

In this paper, we are especially interested in applying our results to equation (1.1) with $f(x) = x^n$, where n is positive and rational.

The concept of "type I Lyapunov funtion" was first defined by Peterson and Tisdell in [9] for the study of existence of solutions of first order dynamic equations.

We say $V : [0,\infty) \times \mathbb{R}^n \mapsto [0,\infty)$ is a *type I Lyapunov functional* on $[0,\infty) \times \mathbb{R}^n$ when

$$V(t,x) = \sum_{i=1}^{n} \Big(V_i(x_i) + U_i(t) \Big),$$

where each $V_i : \mathbb{R} \mapsto \mathbb{R}$ and $U_i : [0,\infty) \mapsto \mathbb{R}$ are continuously differentiable. Next, we extend the definition of the derivative of a type I Lyapunov function to type I Lyapunov functionals. If V is a type I Lyapunov functional and x is a solution of equation (1.1), then we have

$$\begin{aligned}
[V(t,x)]^{\Delta} &= \sum_{i=1}^{n} \Big(V_i(x_i(t)) + U_i(t) \Big)^{\Delta} \\
&= \int_0^1 \nabla V \left[x(t) + h\mu(t) G(t, x(.)) \right] . G(t, x(.)) dh + \sum_{i=1}^{n} U_i^{\Delta}(t)
\end{aligned}$$

where $\nabla = (\partial/\partial x_1, \cdots, \partial/\partial x_n)$ is the gradient operator. This motivates us to define $\dot{V} : [0,\infty) \times \mathbb{R}^n \mapsto \mathbb{R}$ by

$$\dot{V}(t,x) = [V(t,x)]^{\Delta}.$$

Continuing in the spirit of [9], we have

$$\dot{V}(t,x) = \begin{cases} \sum_{i=1}^{n} \frac{V_i\left(x_i+\mu(t)G_i(t,x(\cdot))\right)-V_i(x_i)}{\mu(t)} + \sum_{i=1}^{n} U_i^{\Delta}(t), & \text{when } \mu(t) \neq 0, \\ \nabla V(x).G(t,x(\cdot)) + \sum_{i=1}^{n} U_i^{\Delta}(t), & \text{when } \mu(t) = 0. \end{cases}$$

We also use a continuous strictly increasing function $W_i : [0,\infty) \mapsto [0,\infty)$ with $W_i(0) = 0$, $W_i(s) > 0$ if $s > 0$ for each $i \in \mathbb{Z}^+$.

2 Calculus on Time Scales with Preliminary Results

An introduction with applications and advances in dynamic equations are given in [5, 6]. In this section, we only mention necessary basic results on time scales. We have two jump operators, namely the *forward jump operator* and the *backward jump operator*

$$\sigma(t) := \inf\{s > t : s \in \mathbb{T}\}, \quad \rho(t) := \sup\{s < t : s \in \mathbb{T}\}$$

for all $t \in \mathbb{T}$, respectively. Therefore, there might be four types of points in a time scale, i.e., $\sigma(t) > t$ (*right-scattered point* t), $\rho(t) < t$ (*left-scattered* point t), $\sigma(t) = t$ (*right-dense* point t), and $\rho(t) = t$ (*left-dense* point t). Also $\mu : \mathbb{T} \mapsto [0,\infty)$ defined by $\mu(t) := \sigma(t) - t$ gives the distance between two points in a time scale.

Assume $x : \mathbb{T} \mapsto \mathbb{R}^n$. Then we define $x^{\Delta}(t)$ to be the vector (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left|[x_i(\sigma(t)) - x_i(s)] - x_i^{\Delta}(t)[\sigma(t) - s]\right| \leq \varepsilon\,|\sigma(t) - s|$$

for all $s \in U$ and for each $i = 1,2,\cdots n$. We call $x^{\Delta}(t)$ the *delta derivative* of $x(t)$ at t, and it turns out that $x^{\Delta}(t) = x'(t)$ if $\mathbb{T} = \mathbb{R}$ and $x^{\Delta}(t) = x(t+1) - x(t)$ if $\mathbb{T} = \mathbb{Z}$. If $G^{\Delta}(t) = g(t)$, then the Cauchy integral is defined by

$$\int_a^t g(s)\Delta s = G(t) - G(a).$$

It can be shown that for each continuous function $f : \mathbb{T} \mapsto \mathbb{R}^n$ at $t \in \mathbb{T}$

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} \quad \text{for right-scattered point } t$$

and if the limit exists

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} \quad \text{for right-dense point } t.$$

The product and quotient rules are given by

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t) \tag{2.1}$$

and

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g^{\sigma}(t)} \quad \text{if} \quad g(t)g^{\sigma}(t) \neq 0.$$

for differentiable functions $f,g:\mathbb{T}\mapsto\mathbb{R}^n$ at $t\in\mathbb{T}$. We also have the following simple useful formula

$$f^{\sigma}(t)=f(t)+\mu(t)f^{\Delta}(t), \quad \text{where} \quad f^{\sigma}=f\circ\sigma. \tag{2.2}$$

We say $f:\mathbb{T}\mapsto\mathbb{R}$ is *rd-continuous* provided f is continuous at each right-dense point $t\in\mathbb{T}$ and whenever $t\in\mathbb{T}$ is left-dense $\lim_{s\to t^-}f(s)$ exists as a finite number.

The following chain rule is due to Poetzsche [10].

Theorem 2.1. *Let $f:\mathbb{R}\mapsto\mathbb{R}$ be continuously differentiable and suppose $g:\mathbb{T}\mapsto\mathbb{R}$ is delta differentiable. Then $f\circ g:\mathbb{T}\mapsto\mathbb{R}$ is delta differentiable and the formula*

$$(f\circ g)^{\Delta}(t)=\left\{\int_0^1 f'\left(g(t)+h\mu(t)g^{\Delta}(t)\right)dh\right\}g^{\Delta}(t) \tag{2.3}$$

holds.

We use the following result [5, Theorem 1.117] to calculate the derivative of the Lyapunov function in further sections. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa}=\mathbb{T}-\{m\}$.

Theorem 2.2. *Let $t_0\in\mathbb{T}^{\kappa}$ and assume $k:\mathbb{T}\times\mathbb{T}^{\kappa}\mapsto\mathbb{R}$ is continuous at (t,t), where $t\in\mathbb{T}^{\kappa}$ with $t>t_0$. Also assume that $k(t,\cdot)$ is rd-continuous on $[t_0,\sigma(t)]$. Suppose for each $\varepsilon>0$ there exists a neighborhood of t, independent U of $\tau\in[t_0,\sigma(t)]$, such that*

$$|k(\sigma(t),\tau)-k(s,\tau)-k^{\Delta}(t,\tau)(\sigma(t)-s)|\le\varepsilon|\sigma(t)-s| \ \textit{for all } s\in U,$$

where k^{Δ} denotes the derivative of k with respect to the first variable. Then

$$g(t):=\int_{t_0}^{t}k(t,\tau)\Delta\tau \ \textit{implies } g^{\Delta}(t)=\int_{t_0}^{t}k^{\Delta}(t,\tau)\Delta\tau+k(\sigma(t),t);$$

and

$$h(t):=\int_{t}^{b}k(t,\tau)\Delta\tau \ \textit{implies } k^{\Delta}(t)=\int_{t}^{b}k^{\Delta}(t,\tau)\Delta\tau-k(\sigma(t),t).$$

We apply the following Cauchy–Schwartz inequality in [5, Theorem 6.15] to prove Theorem 3.8.

Theorem 2.3. *Let $a,b\in\mathbb{T}$. For rd-continuous $f,g:[a,b]\mapsto\mathbb{R}$ we have*

$$\int_a^b|f(t)g(t)|\Delta t\le\sqrt{\left\{\int_a^b|f(t)|^2\Delta t\right\}\left\{\int_a^b|g(t)|^2\Delta t\right\}}.$$

Next, we define an exponential function on a time scale in order to obtain exponential stability of the zero solution of (1.1). We say that $p:\mathbb{T}\mapsto\mathbb{R}$ is *regressive* provided $1+\mu(t)p(t)\neq0$ for all $t\in\mathbb{T}^{\kappa}$. We define the set $\mathcal{R}$ of all regressive and rd-continuous functions. We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by $\mathcal{R}^+=\{p\in\mathcal{R}:1+\mu(t)p(t)>0 \text{ for all } t\in\mathbb{T}^{\kappa}\}$. If $p\in\mathcal{R}$, then the *exponential function* $e_p(t,t_0)$ is for each fixed $t_0\in\mathbb{T}$ the unique solution of the initial value problem

$$x^{\Delta}=p(t)x, \ \ x(t_0)=1$$

on $\mathbb{T}$. Under the addition on $\mathcal{R}$ defined by

$$(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t),\ t \in \mathbb{T}^\kappa$$

is an Abelian group (see [5]), where the additive inverse of p, denoted by $\ominus p$, is defined by

$$(\ominus p)(t) = \frac{-p(t)}{1 + \mu(t)p(t)},\ t \in \mathbb{T}^\kappa.$$

We also define the "circle minus" subtraction $\ominus$ on $\mathcal{R}$ by

$$(p \ominus q)(t) := (p \oplus (\ominus q))(t),\ t \in \mathbb{T}^\kappa. \tag{2.4}$$

Therefore,

$$(p \ominus q)(t) = \frac{p(t) - q(t)}{1 + \mu(t)q(t)},\ t \in \mathbb{T}^\kappa.$$

We use the following properties of the exponential function $e_p(t,s)$ which are proved in Bohner and Peterson [5].

Theorem 2.4. *If $p, q \in \mathcal{R}$, then for $t, s, r, t_0 \in \mathbb{T}$*

1. $e_p(t,t) \equiv 1$ *and* $e_0(t,s) \equiv 1$;
2. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$;
3. $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s) = e_p(s,t)$;
4. $\frac{e_p(t,s)}{e_q(t,s)} = e_{p \ominus q}(t,s)$;
5. $e_p(t,s)e_q(t,s) = e_{p \oplus q}(t,s)$;
6. $e_p(t,r)e_p(r,s) = e_p(t,s)$.

The following properties of the exponential function on $\mathbb{T}$ can be found in [3].

Theorem 2.5. *Let $t_0 \in \mathbb{T}$.*

1. *If $p \in \mathcal{R}^+$, then $e_p(t,t_0) > 0$ for all $t \in \mathbb{T}$.*
2. *If $p \geq 0$, then $e_p(t,t_0) \geq 1$ for all $t \geq t_0$. Therefore, $e_{\ominus p}(t,t_0) \leq 1$ for all $t \geq t_0$.*

Also, it follows from Bernoulli's inequality ([5], Theorem 6.2]) that for any time scale, if the constant $\lambda \in \mathcal{R}^+$, then

$$0 < e_{\ominus\lambda}(t,t_0) \leq \frac{1}{1 + \lambda(t - t_0)}, \quad t \geq t_0.$$

It follows that

$$\lim_{t\to\infty} e_{\ominus\lambda}(t,t_0) = 0.$$

In particular, if $\mathbb{T} = \mathbb{R}$, then $e_{\ominus\lambda}(t,t_0) = e^{-\lambda(t-t_0)}$ and if $\mathbb{T} = \mathbb{Z}^+$, then $e_{\ominus\lambda}(t,t_0) = (1 + \lambda)^{-(t-t_0)}$. For the growth of generalized exponential functions on time scales, see Bodine and Lutz [2].

3 Exponential Stability

In this section, we use a non-negative definite type I Lyapunov functional and establish sufficient conditions to obtain (uniformly) exponentially asymptotically stability of the zero solution of equation (1.1).

Definition 3.1. We say that the zero solution of (1.1) is *exponentially asymptotically stable* on $[0,\infty)$ if there exist a positive constant d, a constant $C \in \mathbb{R}^+$, and an $M > 0$ such that for any solution $x(t,t_0,\phi)$ of (1.1),

$$\|x(t,t_0,x_0)\| \leq C\Big(|\phi|,t_0\Big)(e_{\ominus M}(t,t_0))^d, \quad \text{for all } t \in [t_0,\infty)$$

where $C\Big(|\phi|,t_0\Big)$ is a constant depending on $|\phi|$ and t_0, ϕ is a given continuous and bounded initial function, $||.||$ is the Euclidean norm, and $|\phi| = \sup\{||\phi(t)|| : 0 \leq t \leq t_0\}$. We say that the zero solution of system (1.1) is *uniformly exponentially asymptotically stable* on $[0,\infty)$ if C is independent of t_0.

We make use of the above expression in our examples. Following lemma is mentioned in [4].

Lemma 3.2. *Assume $\phi(t,s)$ is right-dense continuous (rd-continuous) with respect to the second variable and is delta-differentiable with respect to the first variable, and let*

$$V(t,x) = x^2(t) + \int_0^t \phi(t,s)W(|x(s)|)\Delta s.$$

If x is a solution of (1.1), then we have by using (2.2) and Theorem 2.2 that

$$\begin{aligned} \dot{V}(t,x) &= 2x \cdot G(t,x(\cdot)) + \mu(t)G^2(t,x(\cdot)) \\ &\quad + \int_0^t \phi^{\Delta}(t,s)W(|x(s)|)\Delta s + \phi(\sigma(t),t)W(|x(t)|), \end{aligned}$$

where $\phi^{\Delta}(t,s)$ denotes the derivative of ϕ with respect to the first variable.

Definition 3.3. We say that a type I Lyapunov functional $V : [0,\infty) \times \mathbb{R}^n \mapsto [0,\infty)$ is *negative definite* if $V(t,x) \neq 0$ for $x \neq 0, x \in \mathbb{R}^n$, $V(t,x) = 0$ for $x = 0$ and along the solutions of (1.1) we have $\dot{V}(t,x) \leq 0$. If the condition $\dot{V}(t,x) \leq 0$ does not hold for all $(t,x) \in \mathbb{T} \times \mathbb{R}^n$, then the Lyapunov functional is said to be *non-negative definite*.

Theorem 3.4. *Let $D \subset \mathbb{R}^n$ containing the origin. Suppose that there exists a type I Lyapunov functional $V : [0,\infty) \times D \mapsto [0,\infty)$ such that for all $(t,x) \in [0,\infty) \times D$*

$$\lambda_1(t)W_1(|x|) \leq V(t,x) \leq \lambda_2(t)W_2(|x|) + \lambda_2(t)\int_0^t \phi_1(t,s)W_3(|x(s)|)\Delta s \tag{3.1}$$

and

$$\dot{V}(t,x) \leq \frac{-\lambda_3(t)W_4(|x|) - \lambda_3(t)\int_0^t \phi_2(t,s)W_5(|x(s))|\Delta s - L(M \ominus \delta)e_{\ominus\delta}(t,0)}{1 + \mu(t)\frac{\lambda_3(t)}{\lambda_2(t)}}. \tag{3.2}$$

where $\lambda_1(t),\lambda_2(t),\lambda_3(t)$ are positive continuous functions, M,δ are positive constants, L is a nonnegative constant, $\lambda_1(t)$ is nondecreasing and $\phi_i(t,s)\geq 0$ is rd-continuous with respect to the second variable for $0\leq s\leq t<\infty$, $i=1,2$ such that

$$W_2(|x|)-W_4(|x|)+\int_0^t(\phi_1(t,s)W_3(|x|)-\phi_2(t,s)W_5(|x(s)|))\Delta s\leq 0. \tag{3.3}$$

If $\int_0^t\phi_1(t,s)\Delta s\leq B$ for some positive constants B, then the zero solution of equation (1.1) *is exponentially asymptotically stable.*

Proof. Let $M:=\inf_{t\geq 0}\frac{\lambda_3(t)}{\lambda_2(t)}>0$ and set $\delta>M$. Let x be a solution of equation (1.1) with $x(t)=\phi(t)$ for $0\leq t\leq t_0$. Since $M\leq\frac{\lambda_3(t)}{\lambda_2(t)}$, we have

$$\begin{aligned}
[V(t,x(t))e_M(t,t_0)]^\Delta &\overset{(2.1)}{=} \dot V(t,x(t))e_M^\sigma(t,t_0)+MV(t,x(t))e_M(t,t_0)\\
&\overset{(2.2)}{=} \left[\dot V(t,x(t))(1+\mu(t)M)+MV(t,x(t))\right]e_M(t,t_0)\\
&\overset{(3.1)(3.2)}{\leq} \Bigg[-\lambda_3(t)W_4(|x|)-\lambda_3(t)\int_0^t\phi_2(t,s)W_5(|x(s)|)\Delta s-L(M\ominus\delta)e_{\ominus\delta}(t,0)\\
&\quad + \lambda_3(t)W_2(|x|)+\lambda_3(t)\int_0^t\phi_1(t,s)W_3(|x(s)|)\Delta s\Bigg]e_M(t,t_0)\\
&\overset{(3.3)}{\leq} -L(M\ominus\delta)e_{\ominus\delta}(t,0)e_M(t,t_0)\\
&\overset{(2.4)}{=} -L(M\ominus\delta)e_{M\ominus\delta}(t,0)e_M(0,t_0),
\end{aligned}$$

where we used Theorem 2.5 (i), and Theorem 2.4 (ii), (v), and (vi). Integrating both sides from t_0 to t, we have from Theorem 2.4 (i) and Theorem 2.5 (i) and (ii)

$$\begin{aligned}
V(t,x(t))e_M(t,t_0) &\leq V(t_0,\phi)-Le_M(0,t_0)e_{M\ominus\delta}(t,0)+Le_M(0,t_0)e_{M\ominus\delta}(t_0,0)\\
&= V(t_0,\phi)-Le_{\ominus\delta}(t,0)e_M(t,t_0)+Le_{\ominus\delta}(t_0,0)\\
&\leq V(t_0,\phi)+L.
\end{aligned}$$

It follows from Theorem 2.4 (iii) that for all $t\geq t_0$

$$V(t,x(t))\leq(V(t_0,\phi)+L)e_{\ominus M}(t,t_0).$$

From inequality (3.1), we have

$$\begin{aligned}
W_1(|x|) &\leq \frac{1}{\lambda_1(t)}(V(t_0,\phi)+L)e_{\ominus M}(t,t_0)\\
&\leq \frac{1}{\lambda_1(t_0)}\left[\lambda_2(t_0)W_2(|\phi|)+\lambda_2(t_0)W_3(|\phi|)\int_0^{t_0}\phi_1(t_0,s)\Delta s+L\right]e_{\ominus M}(t,t_0),
\end{aligned}$$

where we used the fact Theorem 2.5 (ii) and $\lambda_1(t)$ is nondecreasing. Therefore, we obtain

$$|x|\leq W_1^{-1}\left\{\frac{1}{\lambda_1(t_0)}\left[\lambda_2(t_0)W_2(|\phi|)+\lambda_2(t_0)W_3(|\phi|)\int_0^{t_0}\phi_1(t_0,s)\Delta s+L\right]e_{\ominus M}(t,t_0)\right\}$$

for all $t\geq t_0$. This concludes the proof. □

Remark 3.5. In Theorem 3.4, if $\lambda_i(t) = \lambda_i, i = 1,2,3$ are positive constants, then the trivial solution of (1.1) is uniformly exponentially stable on $[0,\infty)$. The proof of this remark follows from Theorem 3.4 by taking $\delta > \frac{\lambda_3}{\lambda_2}$ and $M = \frac{\lambda_3}{\lambda_2}$.

Next, we provide an example in the form of a theorem as an application to Theorem 3.1.

Remark 3.6. Condition (8) can be easily satisfied if $W_2 = W_4$, $W_3 = W_5$ and with the appropriate growth condition on the functions ϕ_1 and ϕ_2, as the next theorem shows.

Remark 3.7. [4, Theorem 3.2] yields boundedness of solutions. However, Theorem 3.4 yields the exponential stability of the zero solution of (1.1). Obtaining exponential stability requires different conditions from those of [4, Theorem 3.2].

Theorem 3.8. *Suppose $B(t,s)$ is rd-continuous with respect to the second variable and for $1 < \delta$, consider the scalar non-linear Volterra integro-dynamic equation*

$$x^\Delta = a(t)x(t) + e_{\ominus\delta}(t,0)\int_0^t B(t,s)x^{2/3}(s)\Delta s,\ \ t \geq 0,\ x(t) = \phi(t)\ \ for\ \ 0 \leq t \leq t_0, \tag{3.4}$$

where $\phi(t)$ is a given bounded continuous initial function $[0,\infty)$, and a is a continuous function $[0,\infty)$. Suppose there are positive constants ν, β_1, β_2, with $\nu \in (0,1)$ such that

$$\begin{aligned}\Big[2a(t) \quad &+ \quad \mu(t)a^2(t) + e_{\ominus\delta}(t,0)\mu(t)|a(t)|\int_0^t |B(t,s)|\Delta s + e_{\ominus\delta}(t,0)\int_0^t |B(t,s)|\Delta s \\ &+ \quad \nu\int_{\sigma(t)}^\infty e_{\ominus\delta}(u,0)|B(u,t)|\Delta u\Big](1+\mu(t)) \leq -1, \end{aligned} \tag{3.5}$$

$$\left\{-\nu + \frac{2}{3}\Big(1 + \mu(t)|a(t)| + \mu(t)\int_0^t |B(t,s)|\Delta s\Big)\right\}(1+\mu(t)) \leq -\frac{1}{4}, \tag{3.6}$$

$$\int_0^t\int_t^\infty e_{\ominus\delta}(u,0)|B(u,s)|\Delta u\Delta s < \infty,\ \int_0^t |B(t,s)|\Delta s < \infty,$$

$$e_{\ominus\delta}(t,0)|B(t,s)| \geq \nu\int_t^\infty e_{\ominus\delta}(u,0)|B(u,s)|\Delta u, \tag{3.7}$$

and

$$\frac{1}{3}\Big(1 + \mu(t)|a(t)| + \mu(t)\int_0^t |B(t,s)|\Delta s\Big)\int_0^t |B(t,s)|\Delta s \leq -L(1\ominus\delta), \tag{3.8}$$

for some positive constant L, then the zero solution of (3.4) *is uniformly exponentially asymptotically stable.*

Proof. Let

$$V(t,x) = x^2(t) + \nu\int_0^t\int_t^\infty e_{\ominus\delta}(u,0)|B(u,s)|\Delta u x^2(s)\Delta s.$$

Using Theorem 2.2 and Lemma 3.2, we have along the solutions of (3.4) that

$$\begin{aligned}
\dot{V}(t,x) &= 2x(t)\Big(a(t)x(t)+e_{\ominus\delta}(t,0)\int_0^t B(t,s)x^{2/3}(s)\Delta s\Big) \qquad (3.9)\\
&+ \mu(t)\Big(a(t)x(t)+e_{\ominus\delta}(t,0)\int_0^t B(t,s)x^{2/3}(s)\Delta s\Big)^2\\
&- \nu e_{\ominus\delta}(t,0)\int_0^t |B(t,s)|x^2(s)\Delta s+\nu\int_{\sigma(t)}^{\infty} e_{\ominus\delta}(u,0)|B(u,t)|x^2(t)\Delta u\\
&\leq 2a(t)x^2(t)+2e_{\ominus\delta}(t,0)\int_0^t |B(t,s)|\,|x(t)|x^{2/3}(s)\Delta s\\
&+ \mu(t)a^2(t)x^2(t)+2e_{\ominus\delta}(t,0)\mu(t)|a(t)|\int_0^t |B(t,s)|\,|x(t)|x^{2/3}(s)\Delta s\\
&+ \mu(t)\Big(e_{\ominus\delta}(t,0)\int_0^t B(t,s)x^{2/3}(s)\Delta s\Big)^2\\
&+ \nu\int_{\sigma(t)}^{\infty} e_{\ominus\delta}(u,0)|B(u,t)|x^2(t)\Delta u-\nu\int_0^t e_{\ominus\delta}(t,0)|B(t,s)|x^2(s)\Delta s.
\end{aligned}$$

Using the fact that $ab\leq a^2/2+b^2/2$ for any real numbers a and b, we have

$$2e_{\ominus\delta}(t,0)\int_0^t |B(t,s)|\,|x(t)|x^{2/3}(s)\Delta s\leq e_{\ominus\delta}(t,0)\int_0^t |B(t,s)|(x^2(t)+x^{4/3}(s))\Delta s.$$

Also, using Theorem 2.3, and Theorem 2.5 (i), (ii) one obtains

$$\begin{aligned}
\Big(\int_0^t e_{\ominus\delta}(t,0)|B(t,s)|x^{2/3}(s)\Delta s\Big)^2 &= \Big(e_{\ominus\delta}(t,0)\int_0^t |B(t,s)|^{1/2}|B(t,s)|^{1/2}x^{2/3}(s)\Delta s\Big)^2\\
&\leq e_{\ominus\delta}(t,0)\int_0^t |B(t,s)|\Delta s\int_0^t |B(t,s)|x^{4/3}(s)\Delta s.
\end{aligned}$$

A substitution of the above two inequalities into (3.9) yields

$$\begin{aligned}
\dot{V}(t,x) &\leq \Big[2a(t)+\mu(t)a^2(t)+e_{\ominus\delta}(t,0)\mu(t)|a(t)|\int_0^t |B(t,s)|\Delta s\\
&+ e_{\ominus\delta}(t,0)\int_0^t |B(t,s)|\Delta s+\nu\int_{\sigma(t)}^{\infty} e_{\ominus\delta}(u,0)|B(u,t)|\Delta u\Big]x^2(t)\\
&+ \Big[e_{\ominus\delta}(t,0)+e_{\ominus\delta}(t,0)\mu(t)|a(t)|+\mu(t)e_{\ominus\delta}(t,0)\int_0^t |B(t,s)|\Delta s\Big]\int_0^t |B(t,s)|x^{4/3}(s)\Delta s\\
&-\nu\int_0^t e_{\ominus\delta}(t,0)|B(t,s)|x^2(s)\Delta s.
\end{aligned}$$

To further simplify the above inequality we make use of Young's inequality, which says for any two nonnegative real numbers w and z, we have

$$wz\leq \frac{w^e}{e}+\frac{z^f}{f},\ \text{ with } 1/e+1/f=1.$$

Thus, for $e=3/2$ and $f=3$, we get

$$\begin{aligned}
\int_0^t |B(t,s)|x^{4/3}(s)\Delta s &= \int_0^t |B(t,s)|^{1/3}|B(t,s)|^{2/3}x^{4/3}(s)\Delta s\\
&\leq \int_0^t \Big(\frac{|B(t,s)|}{3}+\frac{2}{3}|B(t,s)|x^2(s)\Big)\Delta s.
\end{aligned}$$

Hence, the above inequality simplifies to

$$\begin{aligned}
\dot{V}(t,x) &\leq \Big[2a(t)+\mu(t)a^2(t)+e_{\ominus\delta}(t,0)\mu(t)|a(t)|\int_0^t |B(t,s)|\Delta s \\
&+ e_{\ominus\delta}(t,0)\int_0^t |B(t,s)|\Delta s+\nu\int_{\sigma(t)}^{\infty} e_{\ominus\delta}(u,0)|B(u,t)|\Delta u\Big]x^2(t) \\
&+ \Big[-\nu+\frac{2}{3}\Big(1+\mu(t)|a(t)|+\mu(t)\int_0^t |B(t,s)|\Delta s\Big)\Big]\int_0^t e_{\ominus\delta}(t,0)|B(t,s)|x^2(s)\Delta s \\
&+ \frac{1}{3}\Big(1+\mu(t)|a(t)|+\mu(t)\int_0^t |B(t,s)|\Delta s\Big)\int_0^t |B(t,s)|\Delta s\, e_{\ominus\delta}(t,0).
\end{aligned}$$

Multiplying and dividing the above inequality by $1+\mu(t)$ and then applying conditions (3.5), (3.6) and (3.8), $\dot{V}(t,x)$ reduces to

$$\dot{V}(t,x) \leq \frac{-x^2(t)-\frac{1}{4}\int_0^t |B(t,s)|e_{\ominus\delta}(t,0)x^2(s)\Delta s-L(1\ominus\delta)e_{\ominus\delta}(t,0)}{1+\mu(t)}.$$

By taking $W_1=W_2=W_4=x^2(t)$, $W_3=W_5=x^2(s)$, $\lambda_1=\lambda_2=1$, $\lambda_3=\frac{1}{4}$, $\phi_1(t,s)=\nu\int_t^{\infty} e_{\ominus\delta}(u,0)|B(u,s)|\Delta u$, and $\phi_2(t,s)=e_{\ominus\delta}(t,0)|B(t,s)|$, we see that conditions (3.1) and (3.2) of Theorem 3.4 are satisfied. Next we make sure condition (3.3) holds. Use (3.7) to obtain

$$\begin{aligned}
&W_2(|x|)-W_4(|x|)+\int_0^t (\phi_1(t,s)W_3(|x|)-\phi_2(t,s)W_5(|x(s)|))\,\Delta s \\
&= x^2(t)-x^2(t)+\int_0^t \Big(\nu\int_t^{\infty} e_{\ominus\delta}(u,0)|B(u,s)|\Delta u-e_{\ominus\delta}(t,0)|B(t,s)|\Big)x^2(s)\Delta s \leq 0.
\end{aligned}$$

and hence condition (3.3) is satisfied and an application of Theorem 3.4 and Remark 3.6 yield the results. □

The next theorem is a direct consequence of Theorem 3.4.

Theorem 3.9. *Suppose the hypothesis of Theorem 3.4 hold except the condition λ_1 is nondecreasing is replaced by*

there exists a positive constant $\gamma<M$ such that $\lambda_1(t)\geq e_{\ominus\gamma}(t,0), \forall t\geq t_0\geq 0$,

then the zero solution of (1.1) *is exponentially asymptotically stable.*

Proof. The proof is nearly identical to the proof of Theorem 3.4. It follows from the last inequality in the proof of Theorem 3.4 that

$$\begin{aligned}
||x|| &\leq W_1^{-1}\left\{\frac{1}{\lambda_1(t_0)}\Big[\lambda_2(t_0)W_2(|\phi|)+\lambda_2(t_0)W_3(|\phi|)\int_0^{t_0}\phi_1(t_0,s)\Delta s+L\Big]e_{\ominus M}(t,t_0)\right\} \\
&\leq W_1^{-1}\left\{\Big[\lambda_2(t_0)W_2(|\phi|)+\lambda_2(t_0)W_3(|\phi|)\int_0^{t_0}\phi_1(t_0,s)\Delta s+L\Big]e_{\gamma\ominus M}(t,t_0)\right\}
\end{aligned}$$

for all $t\geq t_0$. This completes the proof. □

Remark 3.10. If $\lambda_2(t)$ is uniformly bounded for all $t \geq 0$ and $\limsup_{t\to\infty} \int_0^t \phi_1(t,s)\Delta s \leq B$, where B is a constant, then the zero solution is uniformly exponentially asymptotically stable.

We assert that Theorem 3.8 can be easily generalized to handle scalar nonlinear Volterra integro-dynamic equations of the form

$$x^{\Delta} = a(t)x(t) + e_{\ominus\delta}(t,0)\int_0^t B(t,s)f(s,x(s))\Delta s,$$

where $|f(t,x(t))| \leq x^{2/3}(t) + M$, for some positive constant M. For the next theorem we consider the scalar Volterra integro-dynamic equation

$$x^{\Delta}(t) = a(t)x(t) + \int_0^t B(t,s)\,f(s,x(s))\Delta s + g(t,x(t)) \tag{3.10}$$

where $t \geq 0$, $x(t) = \phi(t)$ for $0 \leq t \leq t_0$, $\phi(t)$ is a given bounded continuous initial function, $a(t)$ is continuous for $t \geq 0$ and $B(t,s)$ is is right-dense continuous for $0 \leq s \leq t < \infty$. We assume $f(t,x)$ and $g(t,x)$ are continuous in x and t, $f(t,0) = g(t,0) = 0$ and satisfy

$$|g(t,x)| \leq \gamma_1(t) + \gamma_2(t)\,|x(t)|,$$

$$|f(t,x)| \leq \gamma(t)\,|x(t)|,$$

where $\gamma(t)$ and $\gamma_2(t)$ are positive and bounded, and $\gamma_1(t)$ is nonnegative and bounded. For the next theorem we need the identity

$$|x(t)|^{\Delta} = \frac{x(t) + x^{\sigma}(t)}{|x(t)| + |x^{\sigma}(t)|}x^{\Delta}(t).$$

Its proof can be found in [7].

We need to have the following result to prove Theorem 3.12.

Theorem 3.11. *Assume $D \subset \mathbb{R}^n$ containing the origin and there exists a type I Lyapunov functional $V : [0,\infty) \times D \to [0,\infty)$ such that for all $(t,x) \in [0,\infty) \times D$:*

$$\lambda_1\|x\|^p \leq V(t,x), \tag{3.11}$$

$$\dot{V}(t,x) \leq \frac{-\lambda_2 V(t,x) - L(\varepsilon\ominus\delta)e_{\ominus\delta}(t,0)}{1+\varepsilon\mu(t)}; \tag{3.12}$$

where $\lambda_1, \lambda_2, p > 0, \delta > 0, L \geq 0$ are constants and $0 < \varepsilon < \lambda_2$. Then the zero solution of equation (1.1) *is exponentially asymptotically stable.*

Proof. For any initial time $t_0 \geq 0$, let x be the solution of (1.1) with $x(t_0) = \phi$. Define ε such that $0 < \varepsilon < \min\{\lambda_2, \delta\}$. Since $\varepsilon \in \mathcal{R}^+$, $e_\varepsilon(t,0)$ is well defined and positive. We obtain by Theorem 2.4 (vi)

$$\begin{aligned}
[V(t,x(t))e_\varepsilon(t,t_0)]^{\Delta} &\overset{(2.1)}{=} \dot{V}(t,x(t))e_\varepsilon^{\sigma}(t,t_0) + \varepsilon V(t,x(t))e_\varepsilon(t,t_0),\\
&\overset{(3.12)}{\leq} \Big(-\lambda_2 V(t,x(t)) - L(\varepsilon\ominus\delta)e_{\ominus\delta}(t,0)\Big)e_\varepsilon(t,t_0) + \varepsilon V(t,x(t))e_\varepsilon(t,t_0),\\
&= e_\varepsilon(t,t_0)[\varepsilon V(t,x(t)) - \lambda_2 V(t,x(t)) - L(\varepsilon\ominus\delta)e_{\ominus\delta}(t,0)]\\
&\leq -L(\varepsilon\ominus\delta)e_\varepsilon(t,t_0)e_{\ominus\delta}(t,0)\\
&\overset{(2.4)}{=} -L(\varepsilon\ominus\delta)e_{\varepsilon\ominus\delta}(t,0)e_\varepsilon(0,t_0).
\end{aligned}$$

Integrating both sides from t_0 to t, we obtain from Theorem 2.5 (i), (ii)

$$\begin{aligned} V(t,x(t))e_{\varepsilon}(t,t_0) &\leq V(t_0,\phi) - Le_{\varepsilon}(0,t_0)\int_{t_0}^{t}(\varepsilon\ominus\delta)e_{\varepsilon\ominus\delta}(s,0)\Delta s \\ &= V(t_0,\phi) - Le_{\varepsilon}(0,t_0)e_{\varepsilon\ominus\delta}(t,0) + Le_{\varepsilon}(0,t_0)e_{\varepsilon\ominus\delta}(t_0,0) \\ &\leq V(t_0,\phi) + Le_{\varepsilon}(0,t_0)e_{\varepsilon\ominus\delta}(t_0,0) \\ &= V(t_0,\phi) + Le_{\ominus\delta}(t,0) \\ &\leq V(t_0,\phi) + L. \end{aligned}$$

and so by Theorem 2.5 (i) and (ii)

$$V(t,x(t)) \leq [V(t_0,\phi)+L]\, e_{\ominus\varepsilon}(t,t_0).$$

Using (3.11) and Theorem 2.5, we obtain

$$||x|| \leq \{\frac{1}{\lambda_1}\}^{1/p}\left[V(t_0,\phi)+L\right]^{1/p}[e_{\ominus\varepsilon}(t,t_0)]^{1/p} \text{ for all } t \geq t_0.$$

This completes the proof. □

For the next theorem we will need to compute $|x(t)|^{\triangle}$. For such a reference we refer the reader to either [1] or [7].

Theorem 3.12. *Suppose there exist constants $k>1$ and ε,α with $0<\varepsilon<\alpha$ such that*

$$\left[a(t)+\gamma_2(t)+k\int_{\sigma(t)}^{\infty}|B(u,t)|\Delta u\gamma(t)\right](1+\varepsilon\mu(t)) \leq -\alpha < 0$$

where $k = 1+\zeta$ for some $\zeta>0$. Suppose

$$(1+\mu(t)\varepsilon)|B(t,s)| \geq \lambda\int_{t}^{\infty}|B(u,s)|\Delta u$$

where $\lambda \geq \frac{k\alpha}{\zeta}$, $0 \leq s < t \leq u < \infty$,

$$\int_0^{t_0}\int_{t_0}^{\infty}|B(u,s)|\Delta u\gamma(s)\Delta s \leq \rho < \infty \ \text{ for all } t_0 \geq 0,$$

and for some positive constants L and δ with $\delta>\varepsilon$,

$$\gamma_1(t)(1+\varepsilon\mu(t)) \leq -L(\varepsilon\ominus\delta)e_{\ominus\delta}(t,0).$$

Then the zero solution of (3.10) *are exponentially asymptotically stable.*

Proof. Define

$$V(t,x(.)) = |x(t)| + k\int_0^t\int_t^{\infty}|B(u,s)|\Delta u|f(s,x(s))|\Delta s.$$

Along the solutions of (3.10) we have

$$
\begin{aligned}
\dot{V}(t,x) &= \frac{x(t)+x^{\sigma}(t)}{|x(t)|+|x^{\sigma}(t)|}x^{\Delta}(t)+k\int_{\sigma(t)}^{\infty}|B(u,t)|\Delta u|f(t,x(t))|-k\int_0^t|B(t,s)||f(s,x(s))|\Delta s \\
&\leq a(t)|x(t)|+\int_0^t|B(t,s)||f(s,x(s))|\Delta s+|g(t,x(t))| \\
&\quad +k\int_{\sigma(t)}^{\infty}|B(u,t)|\Delta u|f(t,x(t))|-k\int_0^t|B(t,s)||f(s,x(s))|\Delta s \\
&\leq \Big[a(t)+\gamma_2(t)+k\int_{\sigma(t)}^{\infty}|B(u,t)|\Delta u\gamma(t)\Big]|x(t)| \\
&\quad +(1-k)\int_0^t|B(t,s)||f(s,x(s))|\Delta s+\gamma_1(t) \\
&= \Big[a(t)+\gamma_2(t)+k\int_{\sigma(t)}^{\infty}|B(u,t)|\Delta u\gamma(t)\Big]|x(t)|\frac{1+\mu(t)\varepsilon}{1+\mu(t)\varepsilon} \\
&- \zeta(1+\mu(t)\varepsilon)\int_0^t|B(t,s)||f(s,x(s))|\Delta s\frac{1}{1+\mu(t)\varepsilon}+(1+\mu(t)\varepsilon)\gamma_1(t)\frac{1}{1+\mu(t)\varepsilon} \\
&\leq -\alpha|x(t)|\frac{1}{1+\mu(t)\varepsilon}-\zeta\lambda\int_0^t\int_t^{\infty}|B(u,s)|\Delta u|f(s,x(s))|\Delta s\frac{1}{1+\mu(t)\varepsilon}-\frac{L(\varepsilon\ominus\delta)e_{\ominus\delta}(t,0)}{1+\mu(t)\varepsilon} \\
&= -\alpha\Big[|x(t)|+k\int_0^t\int_t^{\infty}|B(u,s)|\Delta u|f(s,x(s))|\Delta s\Big]\frac{1}{1+\mu(t)\varepsilon}-\frac{L(\varepsilon\ominus\delta)e_{\ominus\delta}(t,0)}{1+\mu(t)\varepsilon} \\
&= \frac{-\alpha V(t,x)-L(\varepsilon\ominus\delta)e_{\ominus\delta}(t,0)}{1+\mu(t)\varepsilon}.
\end{aligned}
$$

The results follow from Theorem 3.11. □

Remark 3.13. If $V(t_0,\phi)$ is uniformly bounded, then the zero solution of (3.10) is uniformly exponentially asymptotically stable.

In the next theorem we establish sufficient conditions that guarantee the boundedness of all solutions of the vector Volterra integro-dynamic equation

$$x^{\Delta}=Ax(t)+\int_0^t C(t,s)x(s)\Delta s, \tag{3.13}$$

where $t\geq 0$, $x(t)=\phi(t)$ for $0\leq t\leq t_0$, $\phi(t)$ is a given bounded continuous initial $k\times 1$ vector function. Also, A and $C(t,s)$ are $k\times k$ matrix with $C(t,s)$ being continuous on $\mathbb{T}\times\mathbb{T}$, x is $k\times 1$ vector functions that are continuous for $t\in\mathbb{T}$. If D is a matrix, then $|D|$ means the sum of the absolute values of the elements. For what to follow we write x for $x(t)$.

Theorem 3.14. *Suppose $C^T(t,s)=C(t,s)$. Let I be the $k\times k$ identity matrix. Assume there exist positive constants $L,\nu,\xi,\beta_1,\beta_2,\lambda_3$ and $k\times k$ positive definite constant symmetric matrix B such that*

$$\Big[A^TB+BA+\mu(t)A^TBA\Big]\leq -\xi I, \tag{3.14}$$

$$\Big[-\xi+\int_0^{t_0}|B||C(t,s)|\Delta s+\mu(t)\int_0^t|A^TB||C(t,s)|\Delta s+\nu\int_{\sigma(t)}^{\infty}|C(u,t)|\Delta u\Big](1+\mu(t)\lambda_3)\leq -\beta_1, \tag{3.15}$$

$$\Big[|B| - \nu + \mu(t)\Big(|A^T B| + \int_0^t |C(t,s)|\Delta s\Big)\Big](1+\mu(t)\lambda_3) \le -\beta_2, \tag{3.16}$$

$$|C(t,s)| \ge \nu \int_{\sigma(t)}^{\infty} |C(u,s)|\Delta u, \tag{3.17}$$

$$\int_0^t \int_t^{\infty} |C(u,s)|\Delta u \Delta s < \infty, \ \int_0^t |C(t,s)|\Delta s < \infty,$$

and there exists an $r_1 \in (0,1]$ *such that*

$$r_1 x^T x \le x^T B x \le x^T x. \tag{3.18}$$

Then the zero solution of (3.13) *is exponentially asymptotically stable.*

Proof. Let the matrix B be defined by (3.14) and define

$$V(t,x) = x^T B x + \nu \int_0^t \int_t^{\infty} |C(u,s)|\Delta u x^2(s)\Delta s.$$

Here $x^T x = x^2 = (x_1^2 + x_2^2 + \cdots + x_k^2)$. Using the product rule given in (2.1) we have along the solutions of (3.13) that

$$\begin{aligned} \dot V(t,x) &= (x^\Delta)^T B x + (x^\sigma)^T B x^\Delta - \nu \int_0^t |C(t,s)| x^2(s)\Delta s \\ &+ \nu \int_{\sigma(t)}^{\infty} |C(u,t)|\Delta u x^2 \\ &= (x^\Delta)^T B x + (x + \mu(t) x^\Delta)^T B x^\Delta - \nu \int_0^t |C(t,s)| x^2(s)\Delta s \\ &+ \nu \int_{\sigma(t)}^{\infty} |C(u,t)|\Delta u x^2 \\ &= (x^\Delta)^T B x + x^T B x^\Delta + \mu(t)(x^\Delta)^T B x^\Delta \\ &- \nu \int_0^t |C(t,s)| x^2(s)\Delta s + \nu \int_{\sigma(t)}^{\infty} |C(u,t)|\Delta u x^2 \end{aligned} \tag{3.19}$$

Substituting the right hand side of (3.13) for x^Δ into (3.19) and making use of (3.14) we obtain

$$\begin{aligned} \dot V(t,x) &= \Big[Ax + \int_0^t C(t,s)x(s)\Delta s\Big]^T Bx + x^T B\Big[Ax + \int_0^t C(t,s)x(s)\Delta s\Big] \\ &+ \mu(t)\Big[Ax + \int_0^t C(t,s)x(s)\Delta s\Big]^T B\Big[Ax + \int_0^t C(t,s)x(s)\Delta s\Big] \\ &- \nu \int_0^t |C(t,s)| x^2(s)\Delta s + \nu \int_{\sigma(t)}^{\infty} |C(u,t)|\Delta u x^2. \end{aligned} \tag{3.20}$$

By noting that the right side of (3.20) is scalar and by recalling that B is a symmetric matrix, expression (3.20) simplifies to

$$\begin{aligned} \dot V(t,x) &= x^T\Big(A^T B + BA + \mu(t) A^T BA\Big)x + 2\int_0^t x^T BC(t,s)x(s)\Delta s \\ &+ \mu(t)\Big[2x^T A^T B \int_0^t C(t,s)x(s)\Delta s + \int_0^t x^T(s)C(t,s)\Delta s\, B \int_0^t C(t,s)x(s)\Delta s\Big] \\ &- \nu \int_0^t |C(t,s)| x^2(s)\Delta s + \nu \int_{\sigma(t)}^{\infty} |C(u,t)|\Delta u x^2 \end{aligned}$$

$$
\begin{aligned}
&\leq -\xi x^2 + 2\int_0^t |x^T||B||C(t,s)||x(s)|\Delta s \\
&+ \mu(t)\Big[2\int_0^t |x^T||A^T B||C(t,s)||x(s)|\Delta s \\
&+ \int_0^t x^T(s)C(t,s)\,B\,\Delta s \int_0^t C(t,s)x(s)\Delta s\Big] \\
&- \nu\int_0^t |C(t,s)|x^2(s)\Delta s + \nu\int_{\sigma(t)}^{\infty} |C(u,t)|\Delta u x^2. \qquad (3.21)
\end{aligned}
$$

Next, we perform some calculations to simplify inequality (3.21).

$$
2\int_0^t |x^T||B||C(t,s)||x(s)|\Delta s \leq \int_0^t |B||C(t,s)|(x^2 + x^2(s))\Delta s,
$$

$$
2\int_0^t |x^T||A^T B||C(t,s)||x(s)|\Delta s \leq \int_0^t |A^T B||C(t,s)|(x^2 + x^2(s))\Delta s.
$$

Finally,

$$
\begin{aligned}
&\int_0^t x^T(s)C(t,s)\Delta s\, B\, \int_0^t C(t,s)x(s)\Delta s \\
&\leq |B|\,|\int_0^t x^T(s)C(t,s)\Delta s||\int_0^t C(t,s)x(s)\Delta s| \\
&\leq |B|\Big(\int_0^t x^T(s)C(t,s)\Delta s\Big)^2\Big/2 + |B|\Big(\int_0^t C(t,s)x(s)\Delta s\Big)^2\Big/2 \\
&= |B|\Big(\int_0^t C(t,s)x(s)\Delta s\Big)^2 \\
&= |B|\Big(\int_0^t |C(t,s)|^{\frac{1}{2}}|C(t,s)|^{\frac{1}{2}}|x(s)|\Delta s\Big)^2 \\
&\leq |B|\int_0^t |C(t,s)|\Delta s \int_0^t |C(t,s)|x^2(s)\Delta s.
\end{aligned}
$$

A substitution of the above inequalities into (3.21) yields

$$
\begin{aligned}
\dot{V}(t,x) \quad &\leq \quad \Big[-\xi + \int_0^t |B||C(t,s)|\Delta s + \mu(t)\int_0^t |A^T B||C(t,s)|\Delta s + \nu\int_{\sigma(t)}^{\infty} |C(u,t)|\Delta u\Big]x^2 \\
&+ \quad \Big[|B| - \nu + \mu(t)\Big(|A^T B| + |B|\int_0^t |C(t,s)|\Delta s\Big)\Big]\int_0^t |C(t,s)|x^2(s)\Delta s.
\end{aligned}
$$

Multiplying and dividing the above inequality by $1+\mu(t)\lambda_3$ and then applying conditions (3.15) and (3.16), $\dot{V}(t,x)$ reduces to

$$
\dot{V}(t,x) \quad \leq \quad \frac{-\beta_1 x^2 - \beta_2\int_0^t |C(t,s)|x^2(s)\Delta s}{1+\mu(t)\lambda_3},
$$

By taking $W_1 = r_1 x^T x, W_2 = x^T B x,\ W_4 = x^T x,\ W_3 = W_5 = x^2(s),\ \lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \min\{\beta_1, \beta_2\},\ \phi_1(t,s) = \nu\int_t^{\infty} |C(u,s)|\Delta u$, and $\phi_2(t,s) = |C(t,s)|$, we see that conditions

(3.1) and (3.2) of Theorem 3.4 are satisfied. Next we make sure condition (3.3) hold. Using (3.17) and (3.18) we obtain

$$\begin{aligned} & W_2(|x|) - W_4(|x|) + \int_0^t (\phi_1(t,s)W_3(|x|) - \phi_2(t,s)W_5(|x(s)|))\,\Delta s \\ = \;& x^T Bx - x^T x + \int_0^t \Big(\nu \int_t^\infty |C(u,s)|\Delta u - |C(t,s)|\Big) x^2(s)\Delta s \le 0. \end{aligned}$$

Thus condition (3.3) is satisfied with $\gamma = 0$. An application of Theorem 3.4 yields the results. □

Remark 3.15. It is worth mentioning that Theorem 4.3 is new when $\mathbb{T} = \mathbb{R}$.

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