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FOURIER INTEGRAL OPERATORS OF INFINITE ORDER AND APPLICATIONS TO SG-HYPERBOLIC EQUATIONS

By

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Abstract. In this work, we develop a global calculus for a class of Fourier integral operators with symbols $a(x,\xi)$ having exponential growth in $\boldsymbol{R}^{2n}_{x,\xi}$. The functional frame is given by the spaces of type S of Gelfand and Shilov. As an application, we construct a parametrix and prove the existence of a solution for the Cauchy problem associated to SG-hyperbolic operators with one characteristic of constant multiplicity.

Introduction

In this paper we consider some classes of symbols $a(x,\xi)$ of infinite order, i.e. growing exponentially at infinity together with their derivatives, and we investigate the related Fourier integral operators. Pseudodifferential and Fourier integral operators of infinite order have been studied by L. Cattabriga and D. Mari in [2] and by L. Cattabriga and L. Zanghirati in [3], [4], with applications to hyperbolic Cauchy problems in the Gevrey classes. The operators under consideration in [2], [3], [4] have characteristics of constant multiplicity, without Levi conditions or with Gevrey-Levi conditions on the lower order terms. For related results of well-posedness in the Gevrey classes, see S. Mizohata [21], K. Taniguchi [26], K. Shinkai and K. Taniguchi [25], K. Kajitani and T. Nishitani [15], K. Kajitani and S. Wakabayashi [16]. In our work, we consider symbols having an exponential growth with respect to both the variables x and ξ over all $\boldsymbol{R}^{2n}$. Our aim is to give a suitable tool for studying hyperbolic equations with coefficients and data globally defined in the space variables and obtain results of global existence of the solutions. In particular, we are interested to the Cauchy problem

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for SG-hyperbolic operators. SG-operators were studied in the past by several authors, for example C. Parenti [22], E. Schrohe [24], H. O. Cordes [5]. More recently, S. Coriasco [6], S. Coriasco and P. Panarese [7], S. Coriasco and L. Rodino [8], have investigated the Cauchy problem for SG-hyperbolic operators with characteristics of constant multiplicities, proving results of well-posedness in the frame of the Schwartz spaces $\mathscr{S}, \mathscr{S}'$ under a so-called SG-Levi condition. In Section 1, we introduce some spaces of functions which are well known in the literature as spaces of type S, see I. M. Gelfand and G. E. Shilov [9]. Such spaces represent a global version of the Gevrey classes and seem to be the natural domains of our operators. In Sections 2, 3, we define some classes of symbols of infinite order and develop the calculus for the related Fourier integral operators proving a composition theorem. Finally, in Section 4, we consider the Cauchy problem for a SG-operator with one characteristic of constant multiplicity and prove the existence of a solution by means of the construction of a parametrix, expressed as Fourier integral operator of infinite order in our classes. The conditions we assume on the lower order terms are weaker than the SG-Levi condition in [6], [7], [8].

1 Spaces of Functions

In this section we give some basic results concerning the spaces of functions we will deal with in the paper. We refer to [9], [10], [19] for proofs and details. We will denote by $\boldsymbol{Z}_+$ the set of all positive integers and by $\boldsymbol{N}$ the set $\boldsymbol{Z}_+ \cup \{0\}$. Let θ be a positive real number, $\theta > 1$ and let $A, B \in \boldsymbol{Z}_+$.

Definition 1.1. *We denote by $S_{\theta,A}^{\theta,B}(\boldsymbol{R}^n)$ the space of all functions u in $C^\infty(\boldsymbol{R}^n)$ such that*

$$\sup_{\alpha,\beta \in \boldsymbol{N}^n} \sup_{x \in \boldsymbol{R}^n} A^{-|\alpha|} B^{-|\beta|} (\alpha!\beta!)^{-\theta} |x^\alpha \partial_x^\beta u(x)| < +\infty.$$

We set

$$S_\theta^\theta(\boldsymbol{R}^n) = \bigcup_{A,B \in \boldsymbol{Z}_+} S_{\theta,A}^{\theta,B}(\boldsymbol{R}^n).$$

Proposition 1.2. *$S_{\theta,A}^{\theta,B}(\boldsymbol{R}^n)$ is a Banach space endowed with the norm*

$$\|u\|_{A,B,n} = \sup_{\alpha,\beta \in \boldsymbol{N}^n} \sup_{x \in \boldsymbol{R}^n} A^{-|\alpha|} B^{-|\beta|} (\alpha!\beta!)^{-\theta} |x^\alpha \partial_x^\beta u(x)|. \tag{1.1}$$

By Proposition 1.2, we can give to $S_\theta^\theta(\boldsymbol{R}^n)$ the topology of inductive limit of an increasing sequence of Banach spaces. We remark that this topology is equivalent to the one given in [9]. It is useful to give another characterization of the space $S_\theta^\theta(\boldsymbol{R}^n)$, providing another equivalent topology to $S_\theta^\theta(\boldsymbol{R}^n)$, cf. the proof of Theorem 2.9 below.

Proposition 1.3. *$S_\theta^\theta(\boldsymbol{R}^n)$ is the space of all functions $u \in C^\infty(\boldsymbol{R}^n)$ such that*

$$\sup_{\beta \in \boldsymbol{N}^n} \sup_{x \in \boldsymbol{R}^n} B^{-|\beta|}(\beta!)^{-\theta} e^{L|x|^{1/\theta}} |\partial_x^\beta u(x)| < +\infty$$

for some positive B, L.

Proposition 1.4. *The following statements hold:*
(i) *$S_\theta^\theta(\boldsymbol{R}^n)$ is closed under the differentiation;*
(ii) *$S_\theta^\theta(\boldsymbol{R}^n)$ is a nuclear space.*

Remark 1. *We have*

$$G_0^\theta(\boldsymbol{R}^n) \subset S_\theta^\theta(\boldsymbol{R}^n) \subset G^\theta(\boldsymbol{R}^n),$$

where we denote by $G^\theta(\boldsymbol{R}^n)$ the space of all functions $u \in C^\infty(\boldsymbol{R}^n)$ such that, for every compact subset $K \subset \boldsymbol{R}^n$

$$\sup_{\beta \in \boldsymbol{N}^n} B^{-|\beta|}(\beta!)^{-\theta} \sup_{x \in K} |\partial_x^\beta u(x)| < +\infty$$

for some $B = B(K) > 0$ and by $G_0^\theta(\boldsymbol{R}^n)$ the space of all functions of $G^\theta(\boldsymbol{R}^n)$ with compact support.

We shall denote by $S_\theta^{\theta\prime}(\boldsymbol{R}^n)$ the dual space, i.e. the space of all linear continuous forms on $S_\theta^\theta(\boldsymbol{R}^n)$. From (ii) of Proposition 1.4, we deduce the following important result.

Theorem 1.5. *There exists an isomorphism between $\mathscr{L}(S_\theta^\theta(\boldsymbol{R}^n), S_\theta^{\theta\prime}(\boldsymbol{R}^n))$, space of all linear continuous maps from $S_\theta^\theta(\boldsymbol{R}^n)$ to $S_\theta^{\theta\prime}(\boldsymbol{R}^n)$, and $S_\theta^{\theta\prime}(\boldsymbol{R}^{2n})$, which associates to every $T \in \mathscr{L}(S_\theta^\theta(\boldsymbol{R}^n), S_\theta^{\theta\prime}(\boldsymbol{R}^n))$ a form $K_T \in S_\theta^{\theta\prime}(\boldsymbol{R}^{2n})$ such that*

$$\langle Tu, v \rangle = \langle K_T, v \otimes u \rangle$$

for every $u, v \in S_\theta^\theta(\boldsymbol{R}^n)$. K_T is called the kernel of T.

Finally we give a result concerning the action of the Fourier transformation on $S_\theta^\theta(\boldsymbol{R}^n)$.

Proposition 1.6. *The Fourier transformation is an automorphism of* $S_\theta^\theta(\boldsymbol{R}^n)$ *and it can be extended to an automorphism of* $S_\theta^{\theta\prime}(\boldsymbol{R}^n)$.

2 Symbol Classes and Fourier Integral Operators of Infinite Order

In the following we will use the following notations:

$$\langle x\rangle = (1+|x|^2)^{1/2} \quad \text{for } x \in \boldsymbol{R}^n$$

$$\nabla_x \varphi = \left(\frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_n}\right)$$

$$D_t = -i\partial_t$$

$$D_x^\alpha = D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n} \quad \text{for all } \alpha \in \boldsymbol{N}^n, x \in \boldsymbol{R}^n, \text{ where } D_{x_h} = -i\partial_{x_h},\ h = 1,\dots,n.$$

Given two complex-valued functions f, g, we will use the notation $f \simeq g$ to mean that there exists a constant $C > 0$ such that

$$C^{-1}|f(x)| \le |g(x)| \le C|f(x)|.$$

Finally, we will often use the notations $e_1 = (1,0)$, $e_2 = (0,1)$, $e = (1,1)$.

Let μ, ν, θ be real numbers such that $1 < \mu \le \nu$ and $\theta \ge \mu + \nu - 1$.

Definition 2.1. *For every $A > 0$ we denote by $\Gamma_{\mu,\nu,\theta}^\infty(\boldsymbol{R}^{2n}; A)$ the Fréchet space of all functions $a(x,\xi) \in C^\infty(\boldsymbol{R}^{2n})$ satisfying the following condition: for every $\varepsilon > 0$*

$$\|a\|_{A,\varepsilon} = \sup_{\alpha,\beta\in\boldsymbol{N}^n}\ \sup_{(x,\xi)\in\boldsymbol{R}^{2n}} A^{-|\alpha|-|\beta|}(\alpha!)^{-\mu}(\beta!)^{-\nu}\langle\xi\rangle^{|\alpha|}\langle x\rangle^{|\beta|}$$

$$\cdot \exp[-\varepsilon(|x|^{1/\theta}+|\xi|^{1/\theta})]|D_\xi^\alpha D_x^\beta a(x,\xi)| < +\infty$$

endowed with the topology defined by the seminorms $\|\cdot\|_{A,\varepsilon}$, for $\varepsilon > 0$. We set

$$\Gamma_{\mu,\nu,\theta}^\infty(\boldsymbol{R}^{2n}) = \varinjlim_{A\to+\infty} \Gamma_{\mu,\nu,\theta}^\infty(\boldsymbol{R}^{2n}; A)$$

with the topology of inductive limit of Fréchet spaces.

In the sequel we shall also treat symbols of finite order. Let us give a precise definition for such symbols. Let μ, ν be real numbers such that $1 < \mu \le \nu$ and let $m = (m_1, m_2)$ be a vector of $\boldsymbol{R}^2$.

Definition 2.2. *For every $B > 0$ we denote by $\Gamma_{\mu,\nu}^m(\boldsymbol{R}^{2n}; B)$ the Banach space of all functions $a(x,\xi) \in C^\infty(\boldsymbol{R}^{2n})$ such that*

$$\|a\|_B = \sup_{\alpha,\beta\in N^n} \sup_{(x,\xi)\in R^{2n}} B^{-|\alpha|-|\beta|}(\alpha!)^{-\mu}(\beta!)^{-\nu}$$

$$\cdot \langle\xi\rangle^{-m_1+|\alpha|}\langle x\rangle^{-m_2+|\beta|} \cdot |D_\xi^\alpha D_x^\beta a(x,\xi)| < +\infty$$

endowed with the norm $\|\cdot\|_B$ *and define*

$$\Gamma^m_{\mu,\nu}(\boldsymbol{R}^{2n}) = \varinjlim_{B\to+\infty} \Gamma^m_{\mu,\nu}(\boldsymbol{R}^{2n}; B).$$

We observe that $\Gamma^m_{\mu,\nu}(\boldsymbol{R}^{2n}) \subset \Gamma^\infty_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$ for every $m \in \boldsymbol{R}^2$ and for all $\theta \ge \mu+\nu-1$.

Definition 2.3. *A function* $\varphi \in \Gamma^e_{\mu,\nu}(\boldsymbol{R}^{2n})$ *will be called a phase function if it is real-valued and there exists a positive constant* C_φ *such that*

$$C_\varphi^{-1}\langle x\rangle \le \langle\nabla_\xi\varphi\rangle \le C_\varphi\langle x\rangle \tag{2.1}$$

$$C_\varphi^{-1}\langle \xi\rangle \le \langle\nabla_x\varphi\rangle \le C_\varphi\langle \xi\rangle. \tag{2.2}$$

We shall denote by $\mathscr{P}$ *the space of all phase functions.*

Given $a \in \Gamma^\infty_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$ and $\varphi \in \mathscr{P}$, we can consider the Fourier integral operator

$$A_{a,\varphi}u(x) = \int_{\boldsymbol{R}^n} e^{i\varphi(x,\xi)}a(x,\xi)\hat{u}(\xi)\, đ\xi, \quad u \in S^\theta_\theta(\boldsymbol{R}^n) \tag{2.3}$$

where we denote $đ\xi = (2\pi)^{-n}\, d\xi$. A relevant particular case is given by the choice $\varphi(x,\xi) = \langle x,\xi\rangle$, corresponding to the pseudodifferential operator with symbol $a(x,\xi)$ in $\Gamma^\infty_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$

$$Au(x) = \int_{\boldsymbol{R}^n} e^{i\langle x,\xi\rangle}a(x,\xi)\hat{u}(\xi)\, đ\xi.$$

In view of Proposition 1.6 and Definition 2.1, the integral (2.3) is absolutely convergent. To study the operator $A_{a,\varphi}$, we need the following preliminary proposition and lemmas.

Proposition 2.4. *Let* $\varphi \in \Gamma^e_{\mu,\nu}(\boldsymbol{R}^{2n}; B)$ *for some* $B > 0$. *Then, for every* α,β *in* N^n, *there exists a function* $k_{\alpha,\beta}(x,\xi) \in C^\infty(\boldsymbol{R}^{2n})$ *such that*

$$D_\xi^\alpha D_x^\beta e^{i\varphi(x,\xi)} = e^{i\varphi(x,\xi)}k_{\alpha,\beta}(x,\xi)$$

and

$$|k_{\alpha,\beta}(x,\xi)| \le (2^\theta 4B \max\{\|\varphi\|_B, 1\})^{|\alpha|+|\beta|}(|\alpha|!|\beta|!)^\theta \cdot \sum_{h=0}^{\max\{0,|\alpha|-1\}} \frac{\langle x\rangle^{h+1-|\beta|}}{(h!)^\theta} \sum_{k=0}^{\max\{0,|\beta|-1\}} \frac{\langle\xi\rangle^{k+1-|\alpha|}}{(k!)^\theta} \tag{2.4}$$

for all $(x,\xi) \in \boldsymbol{R}^{2n}$.

In the following lemma we collect two well known formulas for factorials and binomial coefficients. The proof is omitted.

LEMMA 2.5. *We have*:

$$(k+j)! \le 2^{k+j} j!k! \quad \forall j,k \in \boldsymbol{N} \tag{2.5}$$

$$\sum_{\substack{\alpha'\le\alpha\\ |\alpha'|=p}} \binom{\alpha}{\alpha'} = \binom{|\alpha|}{p} \quad \forall \alpha \in \boldsymbol{N}^n,\ p \le |\alpha|. \tag{2.6}$$

PROOF OF PROPOSITION 2.4. We argue by induction on $|\alpha+\beta|$. For $\alpha = \beta = 0$, the assertion is trivially verified. For $|\alpha+\beta| \ne 0$, we have

$$\begin{aligned} D_{\xi_i} D_\xi^\alpha D_x^\beta e^{i\varphi(x,\xi)} &= D_\xi^\alpha D_x^\beta [e^{i\varphi(x,\xi)} \partial_{\xi_i}\varphi(x,\xi)] \\ &= e^{i\varphi(x,\xi)} \sum_{\substack{\alpha'\le\alpha\\ \beta'\le\beta}} \binom{\alpha}{\alpha'}\binom{\beta}{\beta'} k_{\alpha-\alpha',\beta-\beta'}(x,\xi) D_\xi^{\alpha'}\partial_{\xi_i} D_x^{\beta'}\varphi(x,\xi). \end{aligned}$$

By the inductive hypothesis, applying (2.5) and observing that $\theta > \max\{\mu,\nu\}$, it follows that

$$\begin{aligned} &\left| \sum_{\substack{\alpha'\le\alpha\\ \beta'\le\beta}} \binom{\alpha}{\alpha'}\binom{\beta}{\beta'} k_{\alpha-\alpha',\beta-\beta'}(x,\xi) D_\xi^{\alpha'}\partial_{\xi_i} D_x^{\beta'}\varphi(x,\xi) \right| \\ &\quad \le \sum_{\substack{\alpha'\le\alpha\\ \beta'\le\beta}} \binom{\alpha}{\alpha'}\binom{\beta}{\beta'} \|\varphi\|_B B^{|\alpha'|+|\beta'|+1} 2^{\mu(|\alpha'|+1)} (\alpha'!)^\mu (\beta'!)^\nu \langle\xi\rangle^{-|\alpha'|}\langle x\rangle^{1-|\beta'|} \\ &\quad \cdot (2^\theta 4B\max\{\|\varphi\|_B,1\})^{|\alpha-\alpha'|+|\beta-\beta'|}(|\alpha-\alpha'|!)^\theta(|\beta-\beta'|!)^\theta \\ &\quad \cdot \sum_{h=0}^{\max\{0,|\alpha-\alpha'|-1\}} \frac{\langle x\rangle^{h+1-|\beta-\beta'|}}{(h!)^\theta} \sum_{k=0}^{\max\{0,|\beta-\beta'|-1\}} \frac{\langle\xi\rangle^{k+1-|\alpha-\alpha'|}}{(k!)^\theta} \end{aligned}$$

$$\leq \frac{1}{4}(2^{\theta}4B\max\{\|\varphi\|_B, 1\})^{|\alpha|+|\beta|+1}$$

$$\cdot\left[\sum_{\alpha'\leq\alpha}\binom{\alpha}{\alpha'}\frac{1}{4^{|\alpha'|}}(|\alpha'|!|\alpha-\alpha'|!)^{\theta}\sum_{h=0}^{\max\{0,|\alpha-\alpha'|-1\}}\frac{\langle x\rangle^{h+2-|\beta|}}{(h!)^{\theta}}\right]$$

$$\cdot\left[\sum_{\beta'\leq\beta}\binom{\beta}{\beta'}\frac{1}{4^{|\beta'|}}(|\beta'|!|\beta-\beta'|!)^{\theta}\sum_{k=0}^{\max\{0,|\beta-\beta'|-1\}}\frac{\langle\xi\rangle^{k+1-|\alpha|}}{(k!)^{\theta}}\right].$$

Now, by (2.6):

$$\sum_{\alpha'\leq\alpha}\binom{\alpha}{\alpha'}\frac{1}{4^{|\alpha'|}}(|\alpha'|!|\alpha-\alpha'|!)^{\theta}\sum_{h=0}^{\max\{0,|\alpha-\alpha'|-1\}}\frac{\langle x\rangle^{h+2-|\beta|}}{(h!)^{\theta}}$$

$$=\sum_{p=0}^{|\alpha|}\binom{|\alpha|}{p}\frac{1}{4^{p}}(p!)^{\theta}(|\alpha|-p)!^{\theta}\sum_{h=0}^{\max\{0,|\alpha|-p-1\}}\frac{\langle x\rangle^{h+2-|\beta|}}{(h!)^{\theta}}$$

$$\leq (|\alpha|!)^{\theta}\sum_{h=0}^{\max\{|\alpha|-1,0\}}\frac{\langle x\rangle^{h+2-|\beta|}}{(h!)^{\theta}}\sum_{p=0}^{\infty}\frac{1}{4^{p}}\leq\frac{4}{3}(|\alpha|+1)!^{\theta}\sum_{h=0}^{\max\{|\alpha|,0\}}\frac{\langle x\rangle^{h+1-|\beta|}}{(h!)^{\theta}}.$$

Analogously, we have

$$\sum_{\beta'\leq\beta}\binom{\beta}{\beta'}\frac{1}{4^{|\beta'|}}(|\beta'|!|\beta-\beta'|!)^{\theta}\sum_{k=0}^{\max\{0,|\beta-\beta'|-1\}}\frac{\langle\xi\rangle^{k+1-|\alpha|}}{(k!)^{\theta}}$$

$$\leq\frac{4}{3}(|\beta|!)^{\theta}\sum_{k=0}^{\max\{0,|\beta|-1\}}\frac{\langle\xi\rangle^{k+1-|\alpha|}}{(k!)^{\theta}}$$

from which (2.4) follows. By estimating similarly $D_{x_i}D_{\xi}^{\alpha}D_x^{\beta}e^{i\varphi(x,\xi)}$, we conclude the proof. □

REMARK 2. *Proposition* 2.4 *implies in particular that for every* $L'>0$ *there exists a constant* $C_{L'}>0$ *such that*

$$|D_x^{\beta}e^{i\varphi(x,\xi)}|\leq C_{L'}^{|\beta|+1}(|\beta|!)^{\theta}\langle x\rangle\langle\xi\rangle e^{L'\langle\xi\rangle^{1/\theta}} \tag{2.7}$$

for all $\beta\in\boldsymbol{N}^n$ *and for all* $(x,\xi)\in\boldsymbol{R}^{2n}$. *Similarly, we have the estimates*

$$|D_{\xi}^{\alpha}D_x^{\beta}e^{i\varphi(x,\xi)}|\leq C_{L'}^{|\alpha|+|\beta|+1}(|\alpha|!|\beta|!)^{\theta}\langle x\rangle\langle\xi\rangle e^{L'(\langle x\rangle^{1/\theta}+\langle\xi\rangle^{1/\theta})} \tag{2.8}$$

for all $\alpha,\beta\in\boldsymbol{N}^n$ *and for all* $(x,\xi)\in\boldsymbol{R}^{2n}$.

Let us set, for $t > 0$,

$$Q_t = \{(x,\xi) \in \boldsymbol{R}^{2n} : \langle x\rangle < t, \langle\xi\rangle < t\}$$
$$Q_t^e = \boldsymbol{R}^{2n} \backslash Q_t.$$

LEMMA 2.6. *For any given $R > 0$ and $\varrho > 1$, we can find a sequence of nonnegative functions $\psi_j(x,\xi) \in C_0^\infty(\boldsymbol{R}^{2n})$ such that $\sum_{j=0}^{\infty} \psi_j(x,\xi) = 1$ on $\boldsymbol{R}^{2n}$,*

$$supp\ \psi_0 \subseteq \overline{Q_{3R}}$$
$$supp\ \psi_j \subseteq \overline{Q_{3R(j+1)^\varrho}} \backslash Q_{2Rj^\varrho}$$

and

$$\sup_{\boldsymbol{R}^{2n}} |D_\xi^\alpha D_x^\beta \psi_j(x,\xi)| \leq C^{|\alpha|+|\beta|+1} (\alpha!)^\mu (\beta!)^\nu [R \sup(j^\varrho, 1)]^{-|\alpha|-|\beta|}$$

for all $\alpha, \beta \in \boldsymbol{N}^n$.

PROOF. Let $\phi \in C_0^\infty(\boldsymbol{R}^{2n})$ such that $0 \leq \phi \leq 1$,

$$\phi(x,\xi) = \begin{cases} 1 & \text{if } (x,\xi) \in Q_2 \\ 0 & \text{if } (x,\xi) \in Q_3^e \end{cases}$$

and

$$\sup_{\boldsymbol{R}^{2n}} |D_\xi^\alpha D_x^\beta \phi(x,\xi)| \leq M^{|\alpha|+|\beta|+1} (\alpha!)^\mu (\beta!)^\nu$$

for all $\alpha, \beta \in \boldsymbol{N}^n$. Let us set

$$g_j(x,\xi) = \phi\left(\frac{x}{Rj^\varrho}, \frac{\xi}{Rj^\varrho}\right), \quad j \geq 1.$$

Finally, we define

$$\psi_0(x,\xi) = g_1(x,\xi)$$
$$\psi_j(x,\xi) = g_{j+1}(x,\xi) - g_j(x,\xi), \quad j \geq 1.$$

This sequence satisfies the conditions of the Lemma. We can assume that ψ_j are nonnegative choosing for example $\phi(x,\xi) = \phi_1(x)\phi_2(\xi)$, where ϕ_1, ϕ_2 are nonnegative functions which radially decrease. $\square$

LEMMA 2.7. *Let $\varphi \in \mathscr{P}$ and denote by $d(x,\xi)$ the function $\langle \nabla_\xi \varphi\rangle^2 - i\Delta_\xi \varphi$. Then, $1/d \in \Gamma_{\mu,\nu}^{(0,-2)}(\boldsymbol{R}^{2n})$.*

PROOF. From (2.1), it follows that

$$|d(x,\xi)| \geq \langle \nabla_\xi \varphi \rangle^2 \geq C_\varphi^{-2} \langle x \rangle^2. \tag{2.9}$$

From (2.9), arguing by induction on the order of the derivatives of d, it is easy to verify the assertion. The details are left to the reader. □

Let us now consider the operator $\mathscr{M}_\xi$ defined by

$$\mathscr{M}_\xi = D(1 - \Delta_\xi) \tag{2.10}$$

where D denotes the multiplication operator by $1/d$. We observe that

$$\mathscr{M}_\xi e^{i\varphi(x,\xi)} = e^{i\varphi(x,\xi)}$$

and that

$${}^t\mathscr{M}_\xi = {}^t(1 - \Delta_\xi)^t D = (1 - \Delta_\xi) D.$$

LEMMA 2.8. *Let $\varphi \in \mathscr{P}$, $a \in \Gamma^\infty_{\mu,\nu,\theta}(\boldsymbol{R}^{2n}; A)$, $u \in S^\theta_\theta(\boldsymbol{R}^n)$ and $\{\psi_j\}_{j \geq 0}$ be a partition of the unity as in Lemma 2.6 with $\varrho = \theta$. Then, there exist positive constants B, C, K, L such that, for every $\varepsilon > 0$ and for every $j, N \in \boldsymbol{N}$, $\alpha, \beta \in \boldsymbol{N}^n$*

$$|D_\xi^\alpha D_x^\beta ({}^t\mathscr{M})^N [\psi_j(x,\xi) a(x,\xi) \hat{u}(\xi)]|$$

$$\leq K \|a\|_{A,\varepsilon} C^{|\alpha|+|\beta|} B^{2N} (|\alpha|!)^\theta (|\beta|!)^\nu ((2N)!)^\theta \langle x \rangle^{-2N} e^{\varepsilon \langle x \rangle^{1/\theta}} e^{-(L-\varepsilon)\langle \xi \rangle^{1/\theta}} \tag{2.11}$$

for all $(x,\xi) \in \boldsymbol{R}^{2n}$.

PROOF. The proof can be obtained by induction on N choosing B, C sufficiently large in the inductive hypothesis. We omit the details for sake of brevity. □

THEOREM 2.9. *Let $\varphi \in \mathscr{P}$. Then, the map $(a, u) \to A_{a,\varphi} u$ is a bilinear and separately continuous map from $\Gamma^\infty_{\mu,\nu,\theta}(\boldsymbol{R}^{2n}) \times S^\theta_\theta(\boldsymbol{R}^n)$ to $S^\theta_\theta(\boldsymbol{R}^n)$ and it can be extended to a bilinear and separately continuous map from $\Gamma^\infty_{\mu,\nu,\theta}(\boldsymbol{R}^{2n}) \times S^{\theta\prime}_\theta(\boldsymbol{R}^n)$ to $S^{\theta\prime}_\theta(\boldsymbol{R}^n)$.*

PROOF. Let us fix $a \in \Gamma^\infty_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$ and show that $u \to A_{a,\varphi} u$ is continuous from $S^\theta_\theta(\boldsymbol{R}^n)$ to itself. Basing on Proposition 1.3, we fix $B \in \boldsymbol{Z}_+$, $L > 0$ and consider the bounded set F determined by $C > 0$

$$\sup_{x \in \boldsymbol{R}^n} e^{L|x|^{1/\theta}} |\partial_x^\beta u(x)| \leq C B^{|\beta|} (\beta!)^\theta$$

for all $u \in F$, $\beta \in \boldsymbol{N}^n$. Estimates of the same type are valid for $\hat{u}$ in view of Proposition 1.6. To prove the continuity with respect to u, we need to show that there exist $A_0, B_0 \in \boldsymbol{Z}_+$ and a constant $C_0 > 0$ such that

$$\sup_{x \in \boldsymbol{R}^n} |x^\alpha D_x^\beta A_{a,\varphi} u(x)| \le C_0 A_0^{|\alpha|} B_0^{|\beta|} (\alpha!\beta!)^\theta \tag{2.12}$$

for all $\alpha, \beta \in \boldsymbol{N}^n$ and for all $u \in F$. Let $\{\psi_j\}_{j \ge 0}$ be a partition of the unity as in Lemma 2.8. We can write, for any fixed $x \in \boldsymbol{R}^n$,

$$x^\alpha D_x^\beta (A_{a,\varphi} u)(x) = x^\alpha D_x^\beta \sum_{j \ge 0} \int_{\boldsymbol{R}^n} e^{i\varphi(x,\xi)} \psi_j(x,\xi) a(x,\xi) \hat{u}(\xi)\, đ\xi. \tag{2.13}$$

Now there exists $j(x) \in \boldsymbol{N}$ such that $2Rj(x)^\theta \le \langle x \rangle < 2R(j(x)+1)^\theta$. Integrating by parts with the operator $\mathscr{M}_\xi$ defined by (2.10), we can decompose the sum in (2.13) as follows:

$$x^\alpha D_x^\beta A_{a,\varphi} u(x) = I_{1\alpha\beta}(x) + I_{2\alpha\beta}(x),$$

where

$$I_{1\alpha\beta}(x) = \sum_{j=0}^{j(x)} x^\alpha D_x^\beta \int_{\boldsymbol{R}^n} e^{i\varphi(x,\xi)} ({}^t\mathscr{M}_\xi)^j [\psi_j(x,\xi) a(x,\xi) \hat{u}(\xi)]\, đ\xi$$

and

$$I_{2\alpha\beta}(x) = \sum_{j > j(x)} x^\alpha D_x^\beta \int_{\boldsymbol{R}^n} e^{i\varphi(x,\xi)} \psi_j(x,\xi) a(x,\xi) \hat{u}(\xi)\, đ\xi.$$

Let us estimate the two terms. By Lemma 2.8 and Remark 2, for every $L' > 0$, there exists $C_{L'} > 0$ such that for every $\varepsilon > 0$

$$\begin{aligned}
|I_{1\alpha\beta}(x)| &\le \sum_{j=0}^{j(x)} \langle x \rangle^{|\alpha|} \sum_{\beta' \le \beta} \binom{\beta}{\beta'} \int_{\boldsymbol{R}^n} |D_x^{\beta-\beta'} e^{i\varphi(x,\xi)}|\, |D_x^{\beta'} ({}^t\mathscr{M}_\xi)^j [\psi_j(x,\xi) a(x,\xi) \hat{u}(\xi)]|\, đ\xi \\
&\le K \|a\|_{A,\varepsilon} \sum_{j=0}^{j(x)} B^{2j} ((2j)!)^\theta \sum_{\beta' \le \beta} \binom{\beta}{\beta'} C^{|\beta'|} C_{L'}^{|\beta-\beta'|} (|\beta|!)^\theta \\
&\quad \cdot \langle x \rangle^{|\alpha|+1-2j} e^{\varepsilon \langle x \rangle^{1/\theta}} \int_{\boldsymbol{R}^n} \langle \xi \rangle e^{-(L-L'-\varepsilon)\langle \xi \rangle^{1/\theta}}\, đ\xi
\end{aligned}$$

where the constants are the same appearing in (2.7) and (2.11). We observe that, on the support of ψ_j and hence on the support of $I_{1\alpha\beta}$, we have

$$\langle x \rangle^{|\alpha|+1} e^{\varepsilon \langle x \rangle^{1/\theta}} \le M_1 A_1^{|\alpha|} (|\alpha|!)^\theta e^{3j}$$

if we choose $\varepsilon < (1/3R)^{1/\theta}$. Furthermore, for $j \le j(x)$, $\langle x \rangle^{-2j} \le (4R^2 j^{2\theta})^{-j}$. Thus, choosing $L' < L$ and $\varepsilon < \min\{L - L', (3R)^{-1/\theta}\}$, it turns out that

$$|I_{1\alpha\beta}(x)| \le M_2 A_1^{|\alpha|} B_1^{|\beta|} \|a\|_{A,\varepsilon} (|\alpha|!|\beta|!)^{\theta} \sum_{j=0}^{\infty} \left(\frac{B^2 e^3}{4R^2}\right)^j \tag{2.14}$$

where the constants $A_1, B_1, M_2 > 0$ are independent of $u \in F$. Choosing R sufficiently large, we obtain the required estimate for $I_{1\alpha\beta}$. Arguing as in the previous case it turns out that there exist positive constants M_3, A_2, L'' such that

$$\begin{aligned} |I_{2\alpha\beta}(x)| &\le \sum_{j > j(x)} \langle x \rangle^{|\alpha|} \sum_{\beta' \le \beta} \binom{\beta}{\beta'} \int_{\boldsymbol{R}^n} |D_x^{\beta-\beta'} e^{i\varphi(x,\xi)}| \, |D_x^{\beta'}[\psi_j(x,\xi) a(x,\xi) \hat{u}(\xi)]| \, đ\xi \\ &\le M_3 \|a\|_{A,\varepsilon} A_2^{|\alpha|+|\beta|} (|\alpha|!|\beta|!)^{\theta} \sum_{j > j(x)} e^{3j} \int_{\boldsymbol{R}^n} \langle \xi \rangle e^{-L'' \langle \xi \rangle^{1/\theta}} \, đ\xi. \end{aligned}$$

Now, for $j > j(x)$, we have $\langle x \rangle < (2R) j^{\theta}$, so $\langle \xi \rangle \ge (2R) j^{\theta}$. Thus

$$|I_{2\alpha\beta}(x)| \le M_4 \|a\|_{A,\varepsilon} A_2^{|\alpha|+|\beta|} (|\alpha|!|\beta|!)^{\theta} \sum_{j \ge 0} \left(\frac{e^3}{e^{L''/2(2R)^{1/\theta}}}\right)^j \tag{2.15}$$

which gives (2.12) for R sufficiently large. Furthermore, from (2.14) and (2.15), we deduce that also the map $a \longrightarrow A_{a,\varphi} u$ is continuous for any fixed $u \in S_\theta^\theta(\boldsymbol{R}^n)$. This concludes the first part of the proof. To prove the second part, we observe that for $u, v \in S_\theta^\theta(\boldsymbol{R}^n)$,

$$\int_{\boldsymbol{R}^n} A_{a,\varphi} u(x) v(x) \, dx = \int_{\boldsymbol{R}^n} \hat{u}(\xi) a_v(\xi) \, d\xi$$

where

$$a_v(\xi) = \int_{\boldsymbol{R}^n} e^{i\varphi(x,\xi)} a(x,\xi) v(x) \, đx.$$

Furthermore, by the same argument of the first part of the proof, it follows that the map $v \longrightarrow a_v$ is linear and continuous from $S_\theta^\theta(\boldsymbol{R}^n)$ to itself. Then, by Proposition 1.6, we can define, for u in $S_\theta^{\theta\prime}(\boldsymbol{R}^n)$,

$$(A_{a,\varphi} u)(v) = \hat{u}(a_v), \quad v \in S_\theta^\theta(\boldsymbol{R}^n).$$

This is a linear and continuous map from $S_\theta^{\theta\prime}(\boldsymbol{R}^n)$ to $S_\theta^{\theta\prime}(\boldsymbol{R}^n)$ whose restriction on $S_\theta^\theta(\boldsymbol{R}^n)$ coincides with $A_{a,\varphi}$ defined by (2.3). It is easy to prove that it is continuous also with respect to a for a fixed u in $S_\theta^{\theta\prime}(\boldsymbol{R}^n)$. $\square$

DEFINITION 2.10. *An operator in $\mathscr{L}(S_\theta^\theta(\boldsymbol{R}^n), S_\theta^\theta(\boldsymbol{R}^n))$ is said to be θ-regularizing if it can be extended to a linear and continuous map from $S_\theta^{\theta\prime}(\boldsymbol{R}^n)$ to $S_\theta^\theta(\boldsymbol{R}^n)$.*

PROPOSITION 2.11. *If $a \in S_\theta^\theta(\boldsymbol{R}^{2n})$, then $A_{a,\varphi}$ is θ-regularizing.*

PROOF. Let us consider the kernel K associated to $A_{a,\varphi}$ given by

$$K(x,y) = \int_{\boldsymbol{R}^n} e^{i(\varphi(x,\xi)-\langle y,\xi\rangle)} a(x,\xi)\, d\!\!\!{}^-\xi.$$

It is sufficient to show that $K \in S_\theta^\theta(\boldsymbol{R}^{2n})$. This will easily imply that $A_{a,\varphi}$ is θ-regularizing. By Proposition 2.4, for every $h,k,\beta,\gamma \in \boldsymbol{N}^n$, we have

$$\begin{aligned}
&x^k y^h D_x^\beta D_y^\gamma K(x,y)\\
&= (-1)^{|\gamma|} x^k y^h \sum_{\beta'\le\beta} \binom{\beta}{\beta'} \int_{\boldsymbol{R}^n} e^{-i\langle y,\xi\rangle} \xi^\gamma D_x^{\beta'} e^{i\varphi(x,\xi)} \cdot D_x^{\beta-\beta'} a(x,\xi)\, d\!\!\!{}^-\xi\\
&= (-1)^{|\gamma|} x^k \sum_{\beta'\le\beta} \binom{\beta}{\beta'} \int_{\boldsymbol{R}^n} e^{-i\langle y,\xi\rangle} D_\xi^h [\xi^\gamma D_x^{\beta'} e^{i\varphi(x,\xi)} \cdot D_x^{\beta-\beta'} a(x,\xi)]\, d\!\!\!{}^-\xi\\
&= (-1)^{|\gamma|} x^k \sum_{\beta'\le\beta} \binom{\beta}{\beta'} \sum_{\substack{h_1+h_2+h_3=h\\ h_1\le\gamma}} \frac{h!}{h_1!h_2!h_3!} \frac{\gamma!}{(\gamma-h_1)!} (-i)^{|h_1|}\\
&\quad \cdot \int_{\boldsymbol{R}^n} e^{i(\varphi(x,\xi)-\langle y,\xi\rangle)} k_{h_2,\beta'}(x,\xi) \xi^{\gamma-h_1} D_\xi^{h_3} D_x^{\beta-\beta'} a(x,\xi)\, d\!\!\!{}^-\xi.
\end{aligned}$$

Hence, by Remark 2, it follows that for every $L' > 0$ there exists $C_{L'} > 0$ such that

$$\begin{aligned}
|x^k y^h D_x^\beta D_y^\gamma K(x,y)| \le& \sum_{\substack{h_1+h_2+h_3=h\\ h_1\le\gamma}} \frac{h!}{h_1!h_2!h_3!} \frac{\gamma!}{(\gamma-h_1)!} (h_2!h_3!)^\theta (\beta!)^\theta C_{L'}^{|\beta|+|h_2|+|h_3|+1}\\
&\cdot \langle x\rangle^{k+1} e^{-(L-L')|x|^{1/\theta}} \int_{\boldsymbol{R}^n} \langle\xi\rangle^{|\gamma|-|h_1|+1} e^{-(L-L')\langle\xi\rangle^{1/\theta}}\, d\!\!\!{}^-\xi.
\end{aligned}$$

Choosing $L' < L$, it turns out that

$$|x^k y^h D_x^\beta D_y^\gamma K(x,y)| \le M_1 M_2^{|h|+|k|} M_3^{|\beta|+|\gamma|} (h!k!\beta!\gamma!)^\theta$$

for some positive constants M_i, $i = 1,2,3$ and for all $(x,y) \in \boldsymbol{R}^{2n}$. Then, K is in $S_\theta^\theta(\boldsymbol{R}^{2n})$. □

To conclude this section, we give a notion of asymptotic expansion for symbols from $\Gamma^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$.

DEFINITION 2.12. *Let $B, C > 0$. We shall denote by $\mathscr{F}\mathscr{S}^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n}; B, C)$ the space of all formal sums $\sum_{j\geq 0} a_j(x,\xi)$ such that $a_j(x,\xi) \in C^{\infty}(\boldsymbol{R}^{2n})$ for all $j \geq 0$ and for every $\varepsilon > 0$*

$$\sup_{j\geq 0}\ \sup_{\alpha,\beta\in \boldsymbol{N}^n}\ \sup_{(x,\xi)\in Q^e_{Bj^{\mu+\nu-1}}} C^{-|\alpha|-|\beta|-2j}(\alpha!)^{-\mu}(\beta!)^{-\nu}(j!)^{-\mu-\nu+1}\langle\xi\rangle^{|\alpha|+j}\langle x\rangle^{|\beta|+j}$$

$$\cdot \exp[-\varepsilon(|x|^{1/\theta} + |\xi|^{1/\theta})]|D^{\alpha}_{\xi}D^{\beta}_{x}a_j(x,\xi)| < +\infty. \tag{2.16}$$

Consider the space $FS^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n}; B, C)$ obtained from $\mathscr{F}\mathscr{S}^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n}; B, C)$ by quotienting by the subspace

$$E = \left\{\sum_{j\geq 0} a_j(x,\xi) \in \mathscr{F}\mathscr{S}^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n}; B, C) : supp(a_j) \subset Q_{Bj^{\mu+\nu-1}}\ \forall j \geq 0\right\}.$$

By abuse of notation, we shall denote the elements of $FS^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n}; B, C)$ by formal sums of the form $\sum_{j\geq 0} a_j(x,\xi)$. The arguments in the following are independent of the choice of representative. We observe that $FS^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n}; B, C)$ is a Fréchet space endowed with the seminorms given by the left-hand side of (2.16), for $\varepsilon > 0$. We set

$$FS^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n}) = \varinjlim_{B,C\to+\infty} FS^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n}; B, C)$$

Every symbol $a \in \Gamma^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$ can be identified with an element $\sum_{j\geq 0} a_j$ of $FS^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$, by setting $a_0 = a$ and $a_j = 0$ for all $j \geq 1$.

DEFINITION 2.13. *We say that two sums $\sum_{j\geq 0} a_j, \sum_{j\geq 0} a'_j$ from $FS^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$ are equivalent (we write $\sum_{j\geq 0} a_j(x,\xi) \sim \sum_{j\geq 0} a'_j(x,\xi)$) if there exist constants $B, C > 0$ such that for all $\varepsilon > 0$*

$$\sup_{N\in \boldsymbol{Z}_+}\ \sup_{\alpha,\beta\in \boldsymbol{N}^n}\ \sup_{(x,\xi)\in Q^e_{BN^{\mu+\nu-1}}} C^{-|\alpha|-|\beta|-2N}(\alpha!)^{-\mu}(\beta!)^{-\nu}(N!)^{-\mu-\nu+1}\langle\xi\rangle^{|\alpha|+N}\langle x\rangle^{|\beta|+N}$$

$$\cdot \exp[-\varepsilon(|x|^{1/\theta} + |\xi|^{1/\theta})]\left|D^{\alpha}_{\xi}D^{\beta}_{x}\sum_{j<N}(a_j - a'_j)\right| < +\infty.$$

THEOREM 2.14. *Given a sum $\sum_{j\geq 0} a_j \in FS^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$, we can find a symbol a in $\Gamma^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$ such that*

$$a \sim \sum_{j \geq 0} a_j \quad in \ FS^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n}).$$

PROOF. Let us consider the functions g_j defined in the proof of Lemma 2.6 with $\varrho = \mu + \nu - 1$ and let us set

$$\varphi_0(x,\xi) = 1$$

$$\varphi_j(x,\xi) = 1 - g_j(x,\xi), \quad j \geq 1.$$

We want to prove that if R is sufficiently large

$$a(x,\xi) = \sum_{j \geq 0} \varphi_j(x,\xi) a_j(x,\xi) \tag{2.17}$$

is well defined as an element of $\Gamma^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$ and $a \sim \sum_{j \geq 0} a_j$ in $FS^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$.

First of all we observe that the sum (2.17) is locally finite so it defines a function $a \in C^{\infty}(\boldsymbol{R}^{2n})$. Consider

$$D^{\alpha}_{\xi} D^{\beta}_{x} a(x,\xi) = \sum_{j \geq 0} \sum_{\substack{\gamma \leq \alpha \\ \delta \leq \beta}} \binom{\alpha}{\gamma} \binom{\beta}{\delta} D^{\beta-\delta}_{x} D^{\alpha-\gamma}_{\xi} a_j(x,\xi) \cdot D^{\delta}_{x} D^{\gamma}_{\xi} \varphi_j(x,\xi).$$

Choosing $R \geq B/2$ where B is the constant in Definition 2.12, we can apply the estimates (2.16) and obtain

$$|D^{\alpha}_{\xi} D^{\beta}_{x} a(x,\xi)| \leq C^{|\alpha|+|\beta|+1} \alpha! \beta! \langle x \rangle^{-|\beta|} \langle \xi \rangle^{-|\alpha|} \exp[\varepsilon(|x|^{1/\theta} + |\xi|^{1/\theta})] \sum_{j \geq 0} H_{j\alpha\beta}(x,\xi)$$

where

$$H_{j\alpha\beta}(x,\xi) = \sum_{\substack{\gamma \leq \alpha \\ \delta \leq \beta}} \frac{[(\alpha-\gamma)!]^{\mu-1} [(\beta-\delta)!]^{\nu-1}}{\gamma! \delta!}$$

$$\cdot \, C^{2j-|\gamma|-|\delta|} (j!)^{\mu+\nu-1} \langle x \rangle^{|\delta|-j} \langle \xi \rangle^{|\gamma|-j} |D^{\delta}_{x} D^{\gamma}_{\xi} \varphi_j(x,\xi)|.$$

In view of the properties of the functions φ_j, we have easily

$$H_{j\alpha\beta}(x,\xi) \leq C_1^{|\alpha|+|\beta|+1} (\alpha!)^{\mu-1} (\beta!)^{\nu-1} \left(\frac{C_2}{R}\right)^j$$

for some positive C_1, C_2 with C_2 independent of R. Enlarging R, it follows that

$$\sum_{j \geq 0}^{\infty} H_{j\alpha\beta}(x,\xi) \leq C_3^{|\alpha|+|\beta|+1} (\alpha!)^{\mu-1} (\beta!)^{\nu-1} \quad \forall (x,\xi) \in \boldsymbol{R}^{2n}$$

from which we deduce that $a \in \Gamma^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$. It remains to prove that $a \sim \sum_{j\geq 0} a_j$. Let us fix $N \in \boldsymbol{N}\backslash\{0\}$. We observe that if $(x,\xi) \in Q^e_{2RN^{\mu+\nu-1}}$, then

$$a(x,\xi) - \sum_{j<N} a_j(x,\xi) = \sum_{j\geq N} \varphi_j(x,\xi)a_j(x,\xi).$$

Thus we have

$$\left|\sum_{j\geq N} D^{\alpha}_{\xi} D^{\beta}_{x}[\varphi_j(x,\xi)a_j(x,\xi)]\right|$$

$$\leq C^{|\alpha|+|\beta|+1}\alpha!\beta!\langle x\rangle^{-|\beta|-N}\langle\xi\rangle^{-|\alpha|-N}\exp[\varepsilon(|x|^{1/\theta}+|\xi|^{1/\theta})]\sum_{j\geq N} H_{jN\alpha\beta}(x,\xi)$$

where

$$H_{jN\alpha\beta}(x,\xi) = \sum_{\substack{\gamma\leq\alpha\\ \delta\leq\beta}} \frac{[(\alpha-\gamma)!]^{\mu-1}[(\beta-\delta)!]^{\nu-1}}{\gamma!\delta!}$$

$$\cdot\, C^{2j-|\gamma|-|\delta|}(j!)^{\mu+\nu-1}\langle x\rangle^{|\delta|+N-j}\langle\xi\rangle^{|\gamma|+N-j}|D^{\delta}_{x}D^{\gamma}_{\xi}\varphi_j(x,\xi)|.$$

Arguing as above we can estimate

$$H_{jN\alpha\beta}(x,\xi) \leq C_4^{2N+|\alpha|+|\beta|+1}(N!)^{\mu+\nu-1}(\alpha!)^{\mu-1}(\beta!)^{\nu-1}$$

and this concludes the proof. □

Proposition 2.15. *Let $\varphi \in \mathscr{P}$ and $a \in \Gamma^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$ such that $a \sim 0$. Then, the operator $A_{a,\varphi}$ is θ-regularizing.*

To prove Proposition 2.15 we need a preliminary result.

Lemma 2.16. *Let $M, r, \varrho, \bar{B}$ be positive numbers, $\varrho \geq 1$. We define*

$$h(\lambda) = \inf_{0\leq N\leq \bar{B}\lambda^{1/\varrho}} \frac{M^{rN}(N!)^r}{\lambda^{rN/\varrho}}, \quad \lambda \in \boldsymbol{R}^+.$$

Then there exist positive constants C, τ such that

$$h(\lambda) \leq Ce^{-\tau\lambda^{1/\varrho}}, \quad \lambda \in \boldsymbol{R}^+.$$

Proof. See [23] for the proof. □

PROOF OF PROPOSITION 2.15. By Definition 2.13, it follows that for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$|D_\xi^\alpha D_x^\beta a(x,\xi)| \le C_\varepsilon C^{|\alpha|+|\beta|}(\alpha!)^\mu(\beta!)^\nu\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|}\exp[\varepsilon(|x|^{1/\theta}+|\xi|^{1/\theta})]$$

$$\cdot \inf_{0\le N\le \bar{B}(\langle\xi\rangle+\langle x\rangle)^{1/(\mu+\nu-1)}} \frac{C^{2N}(N!)^{\mu+\nu-1}}{(\langle\xi\rangle+\langle x\rangle)^N}$$

for every $(x,\xi)\in \boldsymbol{R}^{2n}$, $\alpha,\beta\in \boldsymbol{N}^n$ and for some $\bar{B}, C > 0$ independent of α,β,ε. Applying Lemma 2.16 with $\varrho = r = \mu+\nu-1, \lambda = \langle\xi\rangle+\langle x\rangle$ and taking into account the condition $\theta \ge \mu+\nu-1$, we deduce that for all $\varepsilon > 0$

$$|D_\xi^\alpha D_x^\beta a(x,\xi)| \le C'_\varepsilon C^{|\alpha|+|\beta|}(\alpha!)^\mu(\beta!)^\nu e^{-(\tau-\varepsilon)(|x|^{1/\theta}+|\xi|^{1/\theta})} \tag{2.18}$$

for a certain positive τ. For $0<\varepsilon<\tau$, it follows that $a\in S_\theta^\theta(\boldsymbol{R}^{2n})$. We conclude invoking Proposition 2.11. □

We remark that Definitions 2.12 and 2.13 have an analogous version for symbols of finite order of Definition 2.2 and that all the results of this section hold also for such symbols.

3 The Composition Theorem

We give here our main result which will be applied in the sequel to the solution of the Cauchy problem for certain hyperbolic operators.

THEOREM 3.1. *Let μ,ν,θ be real numbers such that $1<\mu\le\nu$, $\theta\ge\mu+\nu-1$ and let*

$$A_{a,\varphi}u(x) = \int_{\boldsymbol{R}^n} e^{i\varphi(x,\xi)}a(x,\xi)\hat{u}(\xi)\,đ\xi,$$

$$Pu(x) = \int_{\boldsymbol{R}^n} e^{i\langle x,\xi\rangle}p(x,\xi)\hat{u}(\xi)\,đ\xi$$

where $\varphi\in\mathscr{P}$, $a\in\Gamma^\infty_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$, $p\in\Gamma^m_{\mu,\nu}(\boldsymbol{R}^{2n})$ for some $m=(m_1,m_2)\in\boldsymbol{R}^2$. Then $PA_{a,\varphi}$ is, modulo θ-regularizing operators, a Fourier integral operator with phase φ and symbol $q(x,\xi)\in\Gamma^\infty_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$. Furthermore,

$$q(x,\xi)\sim\sum_{j\ge0}q_j(x,\xi) \quad \text{in } FS^\infty_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$$

where

$$q_j(x,\xi) = \sum_{|\alpha|=j}(\alpha!)^{-1}D_y^\alpha((\partial_\xi^\alpha p)(x,\tilde{\nabla}_x\varphi(x,y,\xi))a(y,\xi))_{|_{y=x}} \tag{3.1}$$

and

$$\tilde{\nabla}_x\varphi(x,y,\xi) = \int_0^1 (\nabla_x\varphi)(y+\tau(x-y),\xi)\,d\tau.$$

REMARK 3. *From Theorem* 3.1, *we can recapture the standard composition formula for pseudodifferential operators. Namely, if* $\varphi(x,\xi) = \langle x,\xi\rangle$, *then* $PA_{a,\varphi}$ *is a pseudodifferential operator with symbol* $q(x,\xi) \in \Gamma^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$. *Furthermore,*

$$q(x,\xi) \sim \sum_{\alpha} (\alpha!)^{-1}\partial_\xi^\alpha p(x,\xi) D_x^\alpha a(x,\xi).$$

In particular, if $a \in \Gamma^{m'}_{\mu,\nu}(\boldsymbol{R}^{2n})$, *then* $q(x,\xi) \in \Gamma^{m+m'}_{\mu,\nu}(\boldsymbol{R}^{2n})$ *and* $q(x,\xi) - a(x,\xi)\cdot p(x,\xi)$ *is in* $\Gamma^{m+m'-e}_{\mu,\nu}(\boldsymbol{R}^{2n})$. *We shall give elsewhere full details of this global pseudodifferential calculus, cf. L. Zanghirati* [27], *S. Hashimoto-T. Matsuzawa-Y. Morimoto* [12], *L. Rodino* [23], *T. Aoki* [1]; *the composition formula above is sufficient for the applications in the next section.*

LEMMA 3.2. *Let* q_j, $j \geq 0$ *be defined by* (3.1). *Then,* $\sum_{j\geq 0} q_j \in FS^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$.

PROOF. Let $A, B \in \boldsymbol{Z}_+$ such that $\varphi \in \Gamma^{e}_{\mu,\nu}(\boldsymbol{R}^{2n}; A)$ and $p \in \Gamma^{m}_{\mu,\nu}(\boldsymbol{R}^{2n}; B)$. Let us first show that there exists $C > 0$ such that

$$\begin{aligned}
&|D_\xi^\gamma D_x^\beta [D_y^\alpha(\partial_\xi^\sigma p)(x,\tilde{\nabla}_x\varphi(x,y,\xi))_{|_{y=x}}]| \\
&\quad \leq C(BC_\varphi)^{|\sigma|}(4^{\theta+1}\max\{\|\varphi\|_A, 1\}A^2BC_\varphi(n+1))^{|\alpha|+|\beta|+|\gamma|} \\
&\quad \cdot (|\gamma|+|\sigma|)!^{\mu}(|\alpha|+|\beta|)!^{\nu}\langle\xi\rangle^{m_1-|\gamma|-|\sigma|}\langle x\rangle^{m_2-|\alpha|-|\beta|} \qquad (3.2)
\end{aligned}$$

for all $(x,\xi) \in \boldsymbol{R}^{2n}$, $\alpha,\beta,\gamma,\sigma \in \boldsymbol{N}^n$, where C_φ is the constant appearing in (2.2). We argue by induction on $|\alpha|$. The case $\alpha = 0$ can be treated in turn by induction on $|\beta|+|\gamma|$. The assertion is easily verified if $\beta = \gamma = 0$ for any $\sigma \in \boldsymbol{N}^n$. If $(\beta,\gamma) \neq (0,0)$, fixing the attention on the case $\gamma \neq 0$ and applying (2.5), we have, for some $h \in \{1,\dots,n\}$,

$$\begin{aligned}
&|D_\xi^\gamma D_x^\beta(\partial_\xi^\sigma p)(x,\nabla_x\varphi(x,\xi))| = |D_\xi^{\gamma-e_h} D_x^\beta D_{\xi_h}(\partial_\xi^\sigma p)(x,\nabla_x\varphi(x,\xi))| \\
&\leq \sum_{l=1}^{n}\sum_{\beta'\leq\beta}\binom{\beta}{\beta'}\sum_{\gamma'\leq\gamma-e_h}\binom{\gamma-e_h}{\gamma'}|D_\xi^{\gamma'+e_h}D_x^{\beta'+e_l}\varphi(x,\xi)| \\
&\quad \cdot |D_\xi^{\gamma-\gamma'-e_h}D_x^{\beta-\beta'}(\partial_\xi^{\sigma+e_l}p)(x,\nabla_x\varphi(x,\xi))|
\end{aligned}$$

$$\leq nC(BC_\varphi)^{|\sigma|+1}\langle\xi\rangle^{m_1-|\gamma|-|\sigma|}\langle x\rangle^{m_2-|\beta|}$$

$$\cdot\sum_{\beta'\leq\beta}\binom{\beta}{\beta'}\sum_{\gamma'\leq\gamma-e_h}\binom{\gamma-e_h}{\gamma'}\|\varphi\|_A A^{|\beta'|+|\gamma'|+2}4^{\theta(|\beta'|+|\gamma'|+1)}(|\gamma'|!)^\mu(|\gamma-\gamma'|+|\sigma|)!^\mu$$

$$\cdot(|\beta'|!)^\nu(|\beta-\beta'|)!^\nu(4^{\theta+1}\max\{\|\varphi\|_A,1\}A^2BC_\varphi(n+1))^{|\beta-\beta'|+|\gamma-\gamma'|-1}$$

$$\leq C\frac{n}{4(n+1)}(BC_\varphi)^{|\sigma|}(4^{\theta+1}\max\{\|\varphi\|_A,1\}A^2BC_\varphi(n+1))^{|\beta|+|\gamma|}$$

$$\cdot\langle\xi\rangle^{m_1-|\gamma|-|\sigma|}\langle x\rangle^{m_2-|\beta|}\sum_{\beta'\leq\beta}\binom{\beta}{\beta'}(|\beta'|!)^\nu(|\beta-\beta'|)!^\nu\frac{1}{(n+1)^{|\beta'|}}$$

$$\cdot\sum_{\gamma'\leq\gamma-e_h}\binom{\gamma-e_h}{\gamma'}(|\gamma'|!)^\mu(|\gamma-\gamma'|+|\sigma|)!^\mu\frac{1}{(n+1)^{|\gamma'|}}$$

Now, applying (2.6),

$$\sum_{\gamma'\leq\gamma-e_h}\binom{\gamma-e_h}{\gamma'}(|\gamma'|!)^\mu(|\gamma-\gamma'|+|\sigma|+1)!^\mu\frac{1}{(n+1)^{|\gamma'|}}$$

$$\leq(|\gamma|+|\sigma|)!^\mu\sum_{p=0}^{\infty}\frac{1}{(n+1)^p}=\frac{n+1}{n}(|\gamma|+|\sigma|)!^\mu.$$

In the same way we have

$$\sum_{\beta'\leq\beta}\binom{\beta}{\beta'}(|\beta'|!)^\nu(|\beta-\beta'|)!^\nu\frac{1}{(n+1)^{|\beta'|}}\leq\frac{n+1}{n}(|\beta|!)^\nu\leq 2(|\beta|!)^\nu$$

from which we deduce (3.2) for $\alpha=0$. For what concerns the case $\alpha\neq 0$, from the conditions on p,φ and the inductive hypothesis (3.2), we can directly estimate

$$|D_\xi^\gamma D_x^\beta[D_y^\alpha(\partial_\xi^\sigma p)(x,\tilde\nabla_x\varphi(x,y,\xi))_{|_{y=x}}]|$$

$$=\left|D_\xi^\gamma D_x^\beta\left[D_y^{\alpha-e_h}\sum_{l=1}^n\int_0^1(1-\tau)(D^2_{x_lx_h}\varphi)(y+\tau(x-y),\xi)\,d\tau\right.\right.$$

$$\left.\left.\cdot(\partial_\xi^{\sigma+e_l}p)(x,\tilde\nabla_x\varphi(x,y,\xi))_{|_{y=x}}\right]\right|$$

$$=\left|D_\xi^\gamma D_x^\beta\sum_{l=1}^n\sum_{\alpha'\leq\alpha-e_h}\binom{\alpha-e_h}{\alpha'}\int_0^1(1-\tau)^{|\alpha'|+1}\,d\tau\right.$$

$$\left.\cdot(D_x^{\alpha'+e_h+e_l}\varphi)(x,\xi)D_y^{\alpha-\alpha'-e_h}(\partial_\xi^{\sigma+e_l}p)(x,\tilde\nabla_x\varphi(x,y,\xi)_{|_{y=x}}]\right|.$$

We leave the details to the reader. From (3.2), applying Leibniz formula and the hypothesis on a, we deduce that $\sum_{j\geq 0} q_j \in FS^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$. $\square$

LEMMA 3.3. *Given* $t > 0$, *let*

$$m_t(\eta) = \sum_{q=0}^{\infty} \frac{\eta^q}{(q!)^t}, \quad \eta \geq 0. \tag{3.3}$$

Then, for every $\epsilon > 0$ *there exists a constant* $C = C(t,\epsilon) > 0$ *such that*

$$C^{-1} e^{(t-\epsilon)\eta^{1/t}} \leq m_t(\eta) \leq C e^{(t+\epsilon)\eta^{1/t}} \tag{3.4}$$

for every $\eta \geq 0$.

See [13] for the proof.

In the following we shall also denote for $t, \sigma > 0$, $x \in \boldsymbol{R}^n$,

$$m_{t,\sigma}(x) = m_t(\sigma\langle x\rangle^2).$$

PROOF OF THEOREM 3.1. We start by writing explicitely $PA_{a,\varphi}$ in the form of oscillatory integral. To this end, let $\chi \in S^{\theta}_{\theta}(\boldsymbol{R}^{2n})$ such that $\chi(0,0) = 1$. We have, for any $u \in S^{\theta}_{\theta}(\boldsymbol{R}^n)$,

$$\begin{aligned} PA_{a,\varphi}u(x) &= \lim_{\delta\to 0} \iint e^{i\langle x-y,\eta\rangle} p(x,\eta)\chi(\delta y,\delta\eta)(Au)(y)\, dy đ\eta \\ &= \lim_{\delta\to 0} \iiint e^{i(\varphi(y,\xi)+\langle x-y,\eta\rangle)} p(x,\eta)a(y,\xi)\chi(\delta y,\delta\eta)\hat{u}(\xi)\, dy đ\eta đ\xi \\ &= \lim_{\delta\to 0} \int e^{i\varphi(x,\xi)} \left(\iint e^{i\psi(x,y,\xi,\eta)} p(x,\eta)a(y,\xi)\chi(\delta y,\delta\eta)\, dy đ\eta\right)\hat{u}(\xi)\, đ\xi \end{aligned}$$

where

$$\psi(x,y,\xi,\eta) = \varphi(y,\xi) - \varphi(x,\xi) + \langle x-y,\eta\rangle.$$

Let us prove, sketchily, that the limits exists and does not depend on the choice of χ. First, for every $r \in \boldsymbol{N}$, we have

$$\begin{aligned} &\iint e^{i\psi(x,y,\xi,\eta)} p(x,\eta)a(y,\xi)\chi(\delta y,\delta\eta)\, dy đ\eta \\ &\quad = \iint e^{i(\langle x-y,\eta\rangle-\varphi(x,\xi))}\langle\eta\rangle^{-2r} p(x,\eta)(1-\Delta_y)^r[e^{i\varphi(y,\xi)}a(y,\xi)\chi(\delta y,\delta\eta)]\, dy đ\eta. \end{aligned}$$

If $2r \geq m_1 + n + 1$, then $\langle\eta\rangle^{-2r}p(x,\eta) \in \Gamma_{\mu,\nu}^{(-n-1,m_2)}(\boldsymbol{R}^{2n})$, so it is integrable with respect to η. Furthermore, by Proposition 2.4 and Remark 2,

$$(1-\Delta_y)^r[e^{i\varphi(y,\xi)}a(y,\xi)\chi(\delta y,\delta\eta)] = e^{i\varphi(y,\xi)}b_{r,\delta}(y,\xi,\eta)$$

where $b_{r,\delta}$ is such that for every $\varepsilon > 0$

$$|D_\eta^\beta b_{r,\delta}(y,\xi,\eta)| \leq C_{\varepsilon,r}B^{|\beta|}(\beta!)^\theta \exp[\varepsilon(|y|^{1/\theta} + |\xi|^{1/\theta})] \tag{3.5}$$

for some $C_{\varepsilon,r} > 0$ independent of δ. Moreover, integrating by parts with the operator

$$P_{\sigma,2\theta} = \frac{1}{m_{2\theta,\sigma}(y)}\sum_{q=0}^{\infty}\frac{\sigma^q}{(q!)^{2\theta}}(1-\Delta_\eta)^q$$

we obtain

$$\begin{aligned}
&\iint e^{i\psi(x,y,\xi,\eta)}p(x,\eta)a(y,\xi)\chi(\delta y,\delta\eta)\,dy\,d\!\!\!{}^-\eta \\
&= \sum_{q=0}^{\infty}\frac{\sigma^q}{(q!)^{2\theta}}\iint e^{i(\varphi(y,\xi)-\varphi(x,\xi)-\langle y,\eta\rangle)}\frac{1}{m_{2\theta,\sigma}(y)} \\
&\quad \cdot (1-\Delta_\eta)^q[e^{i\langle x,\eta\rangle}\langle\eta\rangle^{-2r}p(x,\eta)b_{r,\delta}(y,\xi,\eta)]\,dy\,d\!\!\!{}^-\eta
\end{aligned}$$

From (3.5), it follows that

$$(1-\Delta_\eta)^q[e^{i\langle x,\eta\rangle}\langle\eta\rangle^{-2r}p(x,\eta)b_{r,\delta}(y,\xi,\eta)] = e^{i\langle x,\eta\rangle}c_{q,r,\delta}(x,y,\xi,\eta)$$

where $c_{q,r,\delta}(x,y,\xi,\eta)$ is such that

$$|c_{q,r,\delta}(x,y,\xi,\eta)| \leq C_{\varepsilon,r}M_r^q(q!)^{2\theta}\langle x\rangle^{m_2}\langle\eta\rangle^{-n-1}\exp[\langle x\rangle^{1/\theta} + \varepsilon(|y|^{1/\theta} + |\xi|^{1/\theta})] \tag{3.6}$$

with M_r independent of $q,\delta,\varepsilon,\sigma$. Thus, for $\sigma < M_r^{-1}$ and $\varepsilon < \theta\sigma^{1/(2\theta)}$, by applying (3.6), Lemma 3.3 and the standard dominated convergence theorem, it follows that

$$PA_{a,\varphi}u(x) = \int_{\boldsymbol{R}^n} e^{i\varphi(x,\xi)}q(x,\xi)\hat{u}(\xi)\,d\!\!\!{}^-\xi,$$

where

$$\begin{aligned}
q(x,\xi) &= \lim_{\delta\to 0}\iint e^{i\psi(x,y,\xi,\eta)}p(x,\eta)a(y,\xi)\chi(\delta y,\delta\eta)\,dy\,d\!\!\!{}^-\eta \\
&= \sum_{q=0}^{\infty}\frac{\sigma^q}{(q!)^{2\theta}}\iint e^{i(\varphi(y,\xi)-\varphi(x,\xi)-\langle y,\eta\rangle)}\frac{1}{m_{2\theta,\sigma}(y)} \\
&\quad \cdot (1-\Delta_\eta)^q[e^{i\langle x,\eta\rangle}\langle\eta\rangle^{-2r}p(x,\eta)b_{r,0}(y,\xi,\eta)]\,dy\,d\!\!\!{}^-\eta.
\end{aligned} \tag{3.7}$$

We shall write

$$q(x,\xi) = \iint e^{i\psi(x,y,\xi,\eta)} p(x,\eta) a(y,\xi)\, dy đ\eta \tag{3.8}$$

giving to (3.8) the meaning of (3.7). Thanks to the definition above, such integral can be treated in a standard way as an oscillatory integral and we will write it in the sequel in the form (3.8) to simplify the notation.

Let k be a positive number such that $k \in (0,1)$ and let $\chi_k \in G_0^\nu(\boldsymbol{R}^n)$ such that $\chi_k(\sigma) = 1$ if $|\sigma| \leq k/2$ and $\chi_k(\sigma) = 0$ if $|\sigma| \geq k$.

We can decompose

$$q(x,\xi) = q_0(x,\xi) + r_0(x,\xi)$$

where

$$q_0(x,\xi) = \iint e^{i\psi(x,y,\xi,\eta)} \chi_k\left(\frac{y-x}{\langle x\rangle}\right) a(y,\xi) p(x,\eta)\, dy đ\eta$$

and

$$r_0(x,\xi) = \iint e^{i\psi(x,y,\xi,\eta)} \left(1 - \chi_k\left(\frac{y-x}{\langle x\rangle}\right)\right) a(y,\xi) p(x,\eta)\, dy đ\eta.$$

Let now $\varepsilon \in (0,1)$ and $\Phi_\varepsilon \in G_0^\mu(\boldsymbol{R}^n)$ such that $\Phi_\varepsilon(\sigma) = 1$ if $|\sigma| \leq \varepsilon/2$ and $\Phi_\varepsilon(\sigma) = 0$ if $|\sigma| \geq \varepsilon$.

Then

$$q_0(x,\xi) = q_1(x,\xi) + r_1(x,\xi)$$

where

$$q_1(x,\xi) = \iint e^{i\psi(x,y,\xi,\eta)} \chi_k\left(\frac{y-x}{\langle x\rangle}\right) \Phi_\varepsilon\left(\frac{\eta - \nabla_x\varphi(x,\xi)}{\langle\xi\rangle}\right) a(y,\xi) p(x,\eta)\, dy đ\eta$$

and

$$r_1(x,\xi) = \iint e^{i\psi(x,y,\xi,\eta)} \chi_k\left(\frac{y-x}{\langle x\rangle}\right) \left(1 - \Phi_\varepsilon\left(\frac{\eta - \nabla_x\varphi(x,\xi)}{\langle\xi\rangle}\right)\right) a(y,\xi) p(x,\eta)\, dy đ\eta.$$

We observe that

$$\varphi(x,\xi) - \varphi(y,\xi) = \langle x - y, \tilde{\nabla}_x\varphi(x,y,\xi)\rangle.$$

The change of variables

$$z = y - x, \quad \zeta = \eta - \tilde{\nabla}_x\varphi(x,y,\xi)$$

gives

$$q_1(x,\xi) = \iint e^{-i\langle z,\zeta\rangle} \chi_k\left(\frac{z}{\langle x\rangle}\right) \Phi_\varepsilon\left(\frac{\zeta + \tilde{\nabla}_x\varphi(x,x+z,\xi) - \nabla_x\varphi(x,\xi)}{\langle\xi\rangle}\right)$$
$$\times\, a(x+z,\xi) p(x,\zeta + \tilde{\nabla}_x\varphi(x,x+z,\xi))\, dz đ\zeta.$$

If we define

$$b(x,y,\xi,\eta) = \chi_k\left(\frac{y-x}{\langle x\rangle}\right)\Phi_\varepsilon\left(\frac{\eta-\xi+\tilde{\nabla}_x\varphi(x,y,\xi)-\nabla_x\varphi(x,\xi)}{\langle\xi\rangle}\right)$$
$$\times\, a(y,\xi)p(x,\eta-\xi+\tilde{\nabla}_x\varphi(x,y,\xi)),$$

we have

$$q_1(x,\xi) = \iint e^{-i\langle z,\zeta\rangle} b(x,x+z,\xi,\xi+\zeta)\, dz\bar{d}\zeta.$$

A Taylor expansion of b in the last argument near $\zeta = 0$ gives

$$b(x,x+z,\xi,\xi+\zeta) = \sum_{|\alpha|\le j}(\alpha!)^{-1}\zeta^\alpha(\partial_\eta^\alpha b)(x,x+z,\xi,\xi)$$
$$+ (j+1)\sum_{|\alpha|=j+1}(\alpha!)^{-1}\zeta^\alpha\int_0^1(1-t)^j(\partial_\eta^\alpha b)(x,x+z,\xi,\xi+t\zeta)\, dt$$

from which

$$q_1(x,\xi) = \sum_{|\alpha|\le j}(\alpha!)^{-1}(D_y^\alpha\partial_\eta^\alpha b)(x,y,\xi,\eta)\Big|_{\substack{y=x\\ \eta=\xi}}$$
$$+ (j+1)\sum_{|\alpha|=j+1}(\alpha!)^{-1}\int_0^1(1-t)^j r_{\alpha t}(x,\xi)\, dt$$

with

$$r_{\alpha t}(x,\xi) = \iint e^{-i\langle z,\zeta\rangle} D_z^\alpha(\partial_\eta^\alpha b)(x,x+z,\xi,\xi+t\zeta)\, dz\bar{d}\zeta.$$

We observe that χ_k is identically 1 in a neighborhood of $y = x$ and Φ_ε is identically 1 in a neighborhood of $\eta = \xi$, $y = x$, so

$$q_1(x,\xi) = \sum_{|\alpha|\le j}(\alpha!)^{-1}D_y^\alpha((\partial_\xi^\alpha p)(x,\tilde{\nabla}_x\varphi(x,y,\xi))a(y,\xi))_{|_{y=x}}$$
$$+ (j+1)\sum_{|\alpha|=j+1}(\alpha!)^{-1}\int_0^1(1-t)^j r_{\alpha t}(x,\xi)\, dt$$
$$= \sum_{h=0}^{j} q_h(x,\xi) + (j+1)\sum_{|\alpha|=j+1}(\alpha!)^{-1}\int_0^1(1-t)^j r_{\alpha t}(x,\xi)\, dt.$$

Let now $\{\psi_j\}$ be a partition of the unity as in Lemma 2.6, with $\varrho = \mu + \nu - 1$. Then

$$q_1(x,\xi) = \sum_{j\geq 0} \psi_j(x,\xi) \sum_{h=0}^{j} q_h(x,\xi) + r_2(x,\xi)$$

where

$$r_2(x,\xi) = \sum_{j\geq 0} \psi_j(x,\xi) \sum_{|\alpha|=j+1} (\alpha!)^{-1}(j+1) \int_0^1 (1-t)^j r_{\alpha t}(x,\xi)\,dt.$$

By the definition of the ψ_j, it follows that

$$q_1(x,\xi) = \sum_{h\geq 0} q_h(x,\xi)\varphi_h(x,\xi) + r_2(x,\xi)$$

where φ_h are the cut-off functions defined in Theorem 2.14. Finally, we have

$$q(x,\xi) = \sum_{h\geq 0} q_h(x,\xi)\varphi_h(x,\xi) + \sum_{i=0}^{2} r_i(x,\xi).$$

By Proposition 2.11, it is sufficient to show that $r_i \in S_\theta^\theta(\boldsymbol{R}^{2n})$, $i = 0,1,2$, to conclude the proof. Namely, we have to prove that for every $i = 0,1,2$, there exist positive constants A_i, B_i, C_i such that

$$\sup_{(x,\xi)\in\boldsymbol{R}^{2n}} |x^l\xi^{l'} D_\xi^\alpha D_x^\beta r_i(x,\xi)| \leq C_i A_i^{|l|+|l'|} B_i^{|\alpha|+|\beta|} (l!l'!\alpha!\beta!)^\theta \tag{3.9}$$

for all $l, l', \alpha, \beta \in \boldsymbol{N}^n$.

Estimate of r_0.

In order to simplify the notations, we will prove the estimate (3.9) only for $\alpha = \beta = 0$. The case $(\alpha,\beta) \neq (0,0)$ can be treated using the same arguments. Let ψ_j, $j \geq 0$ be as in Lemma 2.6 with $\varrho = \theta$ and $l, l' \in \boldsymbol{N}^n$. We have

$$x^l\xi^{l'} r_0(x,\xi) = \sum_{j\geq 0} \psi_j(x,\xi) x^l \xi^{l'} \iint e^{i\psi(x,y,\xi,\eta)} \left(1 - \chi_k\left(\frac{y-x}{\langle x\rangle}\right)\right) a(y,\xi)p(x,\eta)\,dy\bar{d}\eta.$$

For every fixed $x \in \boldsymbol{R}^n$, there exists $j(x) \in \boldsymbol{N}$ such that $2Rj(x)^\theta \leq \langle x\rangle < 2R(j(x)+1)^\theta$. Then, we can write

$$x^l\xi^{l'} r_0(x,\xi) = J_{1ll'}(x,\xi) + J_{2ll'}(x,\xi)$$

where

$$J_{1ll'}(x,\xi) = \sum_{j=0}^{j(x)} \psi_j(x,\xi)x^l\xi^{l'} \iint e^{i\psi(x,y,\xi,\eta)}\left(1-\chi_k\left(\frac{y-x}{\langle x\rangle}\right)\right)a(y,\xi)p(x,\eta)\,dy\,đ\eta$$

and

$$J_{2ll'}(x,\xi) = \sum_{j>j(x)} \psi_j(x,\xi)x^l\xi^{l'} \iint e^{i\psi(x,y,\xi,\eta)}\left(1-\chi_k\left(\frac{y-x}{\langle x\rangle}\right)\right)a(y,\xi)p(x,\eta)\,dy\,đ\eta.$$

On the support of $1-\chi_k((y-x)/\langle x\rangle)$, we have $|y-x| \geq k/2\langle x\rangle$. Hence, for every $r \in \boldsymbol{N}$,

$$J_{1ll'}(x,\xi) = \sum_{j=0}^{j(x)} \psi_j(x,\xi)x^l\xi^{l'} \sum_{q=0}^{\infty}\frac{\sigma^q}{(q!)^{2\theta}} \iint e^{i\psi(x,y,\xi,\eta)}\left(1-\chi_k\left(\frac{y-x}{\langle x\rangle}\right)\right)a(y,\xi)$$

$$\cdot\, |x-y|^{-2r} m_{2\theta}(\sigma|x-y|^2)^{-1}\Delta_\eta^{q+r}p(x,\eta)\,dy\,đ\eta$$

where

$$m_{2\theta}(\sigma|x-y|^2) = \sum_{q=0}^{\infty}\frac{\sigma^q}{(q!)^{2\theta}}|x-y|^{2q}$$

according to (3.3). Choosing r such that $2r \geq m_1 + n + 1$, it follows that for all $\delta > 0$

$$|a(y,\xi)| m_{2\theta}(\sigma|x-y|^2)^{-1}|\Delta_\eta^{q+r}p(x,\eta)|$$

$$\leq A_\delta B^q (q!)^{2\theta}\langle\eta\rangle^{-n-1}e^{-c\langle x\rangle^{1/\theta}}e^{-(c-\delta)\langle y\rangle^{1/\theta}}e^{\delta\langle\xi\rangle^{1/\theta}}$$

for some constants $A_\delta, B, c > 0$ where B is independent of δ, σ and c is independent of δ. Choosing $\sigma < B^{-1}$ and $\delta < c$, we have

$$|J_{1ll'}(x,\xi)| \leq A_\delta \sum_{j=0}^{j(x)} |\psi_j(x,\xi)|\langle x\rangle^{|l|}\langle\xi\rangle^{|l'|}e^{-c\langle x\rangle^{1/\theta}}e^{\delta\langle\xi\rangle^{1/\theta}}.$$

Now, for $j \leq j(x)$, we have $\langle x\rangle \geq 2Rj^\theta$; moreover, on the support of ψ_j, we have $\langle x\rangle \leq 3R(j+1)^\theta$ and $\langle\xi\rangle \leq 3R(j+1)^\theta$. Then, choosing $\delta < (3R)^{-1/\theta}$,

$$|J_{1ll'}(x,\xi)| \leq A_R B_R^{|l|+|l'|}(|l|!|l'|!)^\theta \sum_{j=0}^{\infty}\frac{e^{3j}}{e^{c(2R)^{1/\theta}j}}$$

which gives (3.9) for R sufficiently large.

Let us estimate $J_{2ll'}(x,\xi)$. Let $\mathscr{M}_y$ be the operator defined by

$$\mathscr{M}_y = \frac{1}{\langle \nabla_y \varphi(y,\xi)\rangle^2 - i\Delta_y\varphi(y,\xi)}(1-\Delta_y).$$

We have

$$J_{2ll'}(x,\xi) = \sum_{j>j(x)} \psi_j(x,\xi)x^l\xi^{l'} \iint e^{i(\varphi(y,\xi)-\varphi(x,\xi)+\langle x,\eta\rangle)}p(x,\eta)$$

$$\cdot ({}^t\mathscr{M}_y)^j\left[e^{-i\langle y,\eta\rangle}a(y,\xi)\left(1-\chi_k\left(\frac{y-x}{\langle x\rangle}\right)\right)\right]\,dy đ\eta.$$

By induction on j, we can easily show that for every $j \in \boldsymbol{N}$

$$({}^t\mathscr{M}_y)^j\left[e^{-i\langle y,\eta\rangle}a(y,\xi)\left(1-\chi_k\left(\frac{y-x}{\langle x\rangle}\right)\right)\right] = e^{-i\langle y,\eta\rangle}\sum_{h=0}^{2j}k_{jh}(x,y,\xi,\eta)$$

where k_{jh} are smooth functions satisfying the following condition: for all $\delta > 0$ there exists a positive constant A_δ such that

$$|D_\xi^\alpha D_x^\beta D_y^\gamma D_\eta^\sigma k_{jh}(x,y,\xi,\eta)| \le A_\delta C^{|\alpha|+|\beta|+|\gamma|+|\sigma|+2j}(|\alpha|!|\beta|!|\gamma|!|\sigma|!)^\theta(2j-h)!^\theta$$

$$\cdot \langle\xi\rangle^{-2j}\langle\eta\rangle^{h-|\sigma|}\exp[\delta(|y|^{1/\theta}+|\xi|^{1/\theta})] \tag{3.10}$$

for all $\alpha,\beta,\gamma,\sigma \in \boldsymbol{N}^n$, $(x,\xi) \in supp(\psi_j)$, $(y,\eta) \in \boldsymbol{R}^{2n}$, and for some $C>0$ independent of δ. Moreover, C is also independent of the parameter R in the expression of the ψ_j. Hence, arguing as for $J_{1ll'}$, we have

$$J_{2ll'}(x,\xi) = \sum_{j>j(x)} \psi_j(x,\xi)x^l\xi^{l'}\sum_{h=0}^{2j}\iint e^{i\psi(x,y,\xi,\eta)}p(x,\eta)k_{jh}(x,y,\xi,\eta)\,dy đ\eta$$

$$= \sum_{j>j(x)} \psi_j(x,\xi)x^l\xi^{l'}\sum_{h=0}^{2j}\sum_{q=0}^{\infty}\frac{\sigma^q}{(q!)^{2\theta}}\iint e^{i\psi(x,y,\xi,\eta)}|x-y|^{-2r(h)}m_{2\theta}(\sigma|x-y|^2)^{-1}$$

$$\cdot \Delta_\eta^{q+r(h)}[p(x,\eta)k_{jh}(x,y,\xi,\eta)]\,dy đ\eta$$

Choosing $r(h) = \min\{N \in \boldsymbol{N} : 2N > h + m_1 + n\}$, in view of (3.10), the integral is convergent. Furthermore, for $j > j(x)$, on the support of ψ_j, we have $\langle\xi\rangle \ge 2Rj^\theta$; hence

$$|J_{2ll'}(x,\xi)| \le A_\delta \sum_{j>j(x)} |\psi_j(x,\xi)|\langle x\rangle^{|l|}\langle\xi\rangle^{|l'|}((2j)!)^\theta\left(\frac{C_3}{2Rj^\theta}\right)^{2j}e^{\delta\langle\xi\rangle^{1/\theta}}$$

with C_3 independent of R. By the conditions on the support of ψ_j we conclude that for R sufficiently large,

$$|J_{2ll'}(x,\xi)| \leq A_R B_R^{|l|+|l'|}(|l|!|l'|!)^{\theta}$$

for all $(x,\xi) \in \boldsymbol{R}^{2n}$. This gives (3.9) for r_0.

Estimate of r_1.

Also for r_1 we can write

$$r_1(x,\xi) = \sum_{j\geq 0} \psi_j(x,\xi) \iint e^{i\psi(x,y,\xi,\eta)} a(y,\xi) p(x,\eta)$$
$$\cdot \chi_k\left(\frac{y-x}{\langle x\rangle}\right)\left(1 - \Phi_\varepsilon\left(\frac{\eta - \nabla_x\varphi(x,\xi)}{\langle\xi\rangle}\right)\right) dy đ\eta$$

with ψ_j as in the estimate of r_0. The change of variables $\varrho = \eta - \nabla_x\varphi(x,\xi)$ gives

$$r_1(x,\xi) = \sum_{j\geq 0} \psi_j(x,\xi) \iint e^{-i\omega(x,y,\xi,\varrho)} a(y,\xi) p(x,\varrho + \nabla_x\varphi(x,\xi))$$
$$\cdot \chi_k\left(\frac{y-x}{\langle x\rangle}\right)\left(1 - \Phi_\varepsilon\left(\frac{\varrho}{\langle\xi\rangle}\right)\right) dy đ\varrho$$

where

$$\omega(x,y,\xi,\varrho) = \langle y - x, \varrho + \nabla_x\varphi(x,\xi)\rangle + \varphi(x,\xi) - \varphi(y,\xi).$$

By the assumption (2.2), there exists $C_3 > 0$ such that

$$|\nabla_y\omega| = |\varrho + \nabla_x\varphi(x,\xi) - \nabla_y\varphi(y,\xi)| \leq C_3(\langle\varrho\rangle + \langle\xi\rangle). \tag{3.11}$$

Moreover, on the support of χ_k we have $|y - x| \leq k\langle x\rangle$, so, if k is sufficiently small, there exists $C_4 > 0$ such that $\langle x + \tau(y-x)\rangle \geq C_4\langle x\rangle$ for every $\tau \in [0,1]$. Hence

$$|\nabla_x\varphi(x,\xi) - \nabla_y\varphi(y,\xi)| \leq \sum_{j,m=1}^{n} \left|\int_0^1 (\partial^2_{x_j x_m}\varphi)(x + \tau(y-x),\xi)(x_m - y_m)\, d\tau\right|$$
$$\leq C_4^{-1}(nB)^2 2^{\theta} k\|\varphi\|_B\langle\xi\rangle \leq C_5 k\langle\xi\rangle$$

with C_5 independent of k. Finally, on the support of $(1 - \Phi_\varepsilon(\varrho/\langle\xi\rangle))$, we have $|\varrho| \geq \varepsilon/2\langle\xi\rangle$, so there exists $C_6 > 0$ such that $|\varrho| \geq C_6\langle\varrho\rangle$. Thus

$$|\nabla_y\omega| \geq |\varrho| - |\nabla_x\varphi - \nabla_y\varphi| \geq C_6\langle\varrho\rangle - C_5 k\langle\xi\rangle$$
$$= \frac{C_6}{2}\langle\varrho\rangle + \left(\frac{C_6\varepsilon}{4} - C_5 k\right)\langle\xi\rangle.$$

Choosing $0 < k < C_6\varepsilon/4C_5$, it follows that there exists $M > 0$ such that

$$M^{-1}(\langle\varrho\rangle + \langle\xi\rangle) \le |\nabla_y\omega| \le M(\langle\varrho\rangle + \langle\xi\rangle).$$

Let U be the operator defined by

$$U = \frac{1}{|\nabla_y\omega|^2}\sum_{k=1}^{n}(i\partial_{y_k}\omega)\partial_{y_k}$$

which satisfies the relation $Ue^{-i\omega} = e^{-i\omega}$. Then, integrating by parts, we have

$$r_1(x,\xi) = \sum_{j\ge0}\psi_j(x,\xi)\iint e^{-i\omega(x,y,\xi,\varrho)}p(x,\varrho+\nabla_x\varphi(x,\xi))\left(1-\Phi_\varepsilon\left(\frac{\varrho}{\langle\xi\rangle}\right)\right)$$
$$\cdot\,({}^tU)^{s(j)}\left[a(y,\xi)\chi_k\left(\frac{y-x}{\langle x\rangle}\right)\right]dy\,đ\varrho$$

where $s(j)$ is a positive integer which will be fixed later depending on j. We want to show that for every $\delta > 0$ there exists $A_\delta > 0$ such that for every $\gamma\in\boldsymbol{N}^n$, $s\in\boldsymbol{N}$,

$$\left|D_y^\gamma({}^tU)^s\left[a(y,\xi)\chi_k\left(\frac{y-x}{\langle x\rangle}\right)\right]\right|$$
$$\le A_\delta C^{|\gamma|+2s}(s+|\gamma|)!^\nu\langle x\rangle^{-s-|\gamma|}(\langle\varrho\rangle+\langle\xi\rangle)^{-s}\exp[\delta(|y|^{1/\theta}+|\xi|^{1/\theta})] \quad (3.12)$$

for every $x,y,\xi\in\boldsymbol{R}^n$ and for some $C>0$ independent of γ,δ,s. We start by observing that if $\gamma\in\boldsymbol{N}^n$, $|\gamma|\ge2$, then $\partial_y^\gamma\omega = -\partial_y^\gamma\varphi$. Moreover, it is easy to prove by induction on $|\gamma|$ that there exists $C_\omega>0$ such that

$$\left|\partial_y^\gamma\frac{1}{|\nabla_y\omega|^2}\right| \le M^2C_\omega^{|\gamma|}(|\gamma|!)^\nu\langle x\rangle^{-|\gamma|}(\langle\varrho\rangle+\langle\xi\rangle)^{-2}$$

for all $x,y,\xi,\varrho\in\boldsymbol{R}^n$, with $\chi_k((y-x)/\langle x\rangle)\ne0$. Using these estimates, we can prove (3.12) by induction on s. Finally we observe that

$$|p(x,\varrho+\nabla_x\varphi(x,\xi))| \le C_p(\langle\varrho\rangle+\langle\xi\rangle)^{m_1}\langle x\rangle^{m_2}.$$

Choosing $s(j) = |m_1|+|m_2|+n+1+j$, we deduce that

$$|p(x,\varrho+\nabla_x\varphi(x,\xi))|\left|1-\Phi_\varepsilon\left(\frac{\varrho}{\langle\xi\rangle}\right)\right|\cdot\left|({}^tU)^{s(j)}\left[a(y,\xi)\chi_k\left(\frac{y-x}{\langle x\rangle}\right)\right]\right|$$
$$\le A(\delta,n)M_1^j\langle x\rangle^{-j}\langle\xi\rangle^{-j}(j!)^\theta\langle y\rangle^{-n-1}\langle\varrho\rangle^{-n-1}\exp[\delta((\sqrt{2}+k)\langle x\rangle)^{1/\theta}]e^{\delta\langle\xi\rangle^{1/\theta}}$$

for some positive M_1 independent of j. Choosing $\delta < (3R(\sqrt{2}+k))^{-1/\theta}$, it turns out that for every $l, l' \in \boldsymbol{N}^n$:

$$|x^l \xi^{l'} r_1(x,\xi)| \leq A(R,n) \sum_{j \geq 0} |\psi_j(x,\xi)| B_R^{|l|+|l'|} (|l|!|l'|!)^\theta (M_1 e^3)^j (j!)^\theta \langle x \rangle^{-j} \langle \xi \rangle^{-j}.$$

We conclude invoking the fact that $\langle x \rangle^{-j} \langle \xi \rangle^{-j} \leq (2R)^{-j} j^{-j\theta}$ on the support of ψ_j and choosing $R > M_1 e^3/2$.

Estimate of r_2.

It remains to prove that $r_2 \in S_\theta^\theta(\boldsymbol{R}^{2n})$. To this end, we begin to prove that for every $\delta > 0$ there exist $A_\delta > 0$ such that

$$\begin{aligned} |D_\xi^\gamma D_x^\beta [D_y^\alpha (\partial_\eta^\alpha b)(x,y,\xi,\xi+t\zeta)]_{|_{y=x+z}}| &\leq A_\delta B^{2|\alpha|+|\beta|+|\gamma|} (\alpha!)^{\mu+\nu} (\beta!\gamma!)^\nu \\ &\cdot \langle \xi \rangle^{m_1-|\alpha|-|\gamma|} \langle x \rangle^{m_2-|\alpha|-|\beta|} \langle \zeta \rangle^{m_1} \exp[\delta(\langle x \rangle^{1/\theta} + \langle z \rangle^{1/\theta} + \langle \xi \rangle^{1/\theta})] \end{aligned} \tag{3.13}$$

for all $\alpha, \beta, \gamma \in \boldsymbol{N}^n$, $(x,\xi) \in \boldsymbol{R}^{2n}$, $(z,\zeta) \in \boldsymbol{R}^{2n}$, $t \in [0,1]$ and for some $B > 0$ independent of $\alpha, \beta, \gamma, \delta$. First of all, we observe that on the support of $\chi_k(z/\langle x \rangle)$, we have $\langle x+z \rangle \geq c\langle x \rangle$ for some positive constant c. By this consideration, it is easy to prove that for all $\delta > 0$ there exists $A_{1\delta} > 0$ such that

$$\begin{aligned} &|D_x^\beta D_\xi^\gamma (D_y^\alpha a)(y,\xi)_{|_{y=x+z}}| \\ &\quad \leq A_{1\delta} B_1^{|\alpha|+|\beta|+|\gamma|} (\gamma!)^\mu (\alpha!\beta!)^\nu \langle \xi \rangle^{-|\gamma|} \langle x \rangle^{-|\alpha|-|\beta|} \exp[\delta(\langle x \rangle^{1/\theta} \\ &\qquad + \langle z \rangle^{1/\theta} + \langle \xi \rangle^{1/\theta})] \end{aligned} \tag{3.14}$$

for all $\alpha, \beta, \gamma \in \boldsymbol{N}^n$ and for all $x, z, \xi \in \boldsymbol{R}^n$ with $\chi_k(z/\langle x \rangle) \neq 0$ with $B_1 > 0$ independent of $\alpha, \beta, \gamma, \delta$. Furthermore, there exists $B_2 > 0$ such that

$$\left| D_x^\beta \left[D_y^\alpha \chi_k \left(\frac{y-x}{\langle x \rangle} \right) \right]_{|_{y=x+z}} \right| \leq B_2^{|\alpha|+|\beta|+1} (\alpha!\beta!)^\nu \langle x \rangle^{-|\alpha|-|\beta|} \tag{3.15}$$

for all $\alpha, \beta \in \boldsymbol{N}^n$ and for all $x, z \in \boldsymbol{R}^n$.

We need an estimate of the derivatives of $D_y^\alpha (\partial_\eta^\alpha p)(x, \tilde{\nabla}_x \varphi(x,y,\xi) + t\zeta)_{|_{y=x+z}}$. We claim that there exist positive constants A_0, B_0 such that

$$\begin{aligned} &|D_\xi^\gamma D_x^\beta [D_y^\alpha (\partial_\eta^\sigma p)(x, \tilde{\nabla}_x \varphi(x,y,\xi) + t\zeta)]| \\ &\quad \leq A_0 (4^\theta (n+1) A_\varphi B_\varphi^2 B_0)^{|\alpha|+|\beta|+|\gamma|} B_0^{|\sigma|} \\ &\quad \cdot (|\gamma| + |\sigma|)!^\mu (|\alpha| + |\beta|)!^\nu \langle \xi \rangle^{m_1-|\gamma|-|\sigma|} \langle x \rangle^{m_2-|\alpha|-|\beta|} \end{aligned} \tag{3.16}$$

for all $\alpha, \beta, \gamma, \sigma \in N^n$. We prove (3.16) by induction on $|\alpha|$. For $\alpha = 0$, we can argue in turn by induction on $|\beta + \gamma|$. If $\beta = \gamma = 0$, we have

$$|(\partial_\eta^\sigma p)(x, \tilde{\nabla}_x \varphi(x, y, \xi) + t\zeta)| \le A_0 B_0^{|\sigma|}(|\sigma|!)^\mu \langle x \rangle^{m_2} \langle t\zeta + \tilde{\nabla}_x \varphi(x, x+z, \xi) \rangle^{m_1 - |\sigma|}.$$

We observe that on the support of $\Phi_\varepsilon((\tilde{\nabla}_x \varphi(x, y, \xi) - \nabla_x \varphi(x, \xi) + t\zeta)/\langle \xi \rangle)$ the following condition holds:

$$|\tilde{\nabla}_x \varphi(x, y, \xi) - \nabla_x \varphi(x, \xi) + t\zeta| \le \varepsilon \langle \xi \rangle$$

from which we deduce that

$$\langle \tilde{\nabla}_x \varphi(x, y, \xi) + t\zeta \rangle \le |\nabla_x \varphi(x, \xi)| + 1 + \varepsilon \langle \xi \rangle \le (1 + \varepsilon + C_\varphi) \langle \xi \rangle$$

where C_φ is the constant appearing in (2.2). Moreover

$$\langle \tilde{\nabla}_x \varphi(x, y, \xi) + t\zeta \rangle \ge \sqrt{2}^{-1}(\langle \nabla_x \varphi \rangle - \varepsilon \langle \xi \rangle) \ge \sqrt{2}^{-1}(C_\varphi^{-1} - \varepsilon) \langle \xi \rangle \ge C' \langle \xi \rangle$$

if we choose ε sufficiently small. Thus, there exist positive constants $\bar{A}_0, \bar{B}_0$ such that

$$|(\partial_\eta^\sigma p)(x, \tilde{\nabla}_x \varphi(x, x+z, \xi) + t\zeta)| \le \bar{A}_0 \bar{B}_0^{|\sigma|}(|\sigma|!)^\mu \langle \xi \rangle^{m_1 - |\sigma|} \langle x \rangle^{m_2}$$

for all $\sigma \in N^n$ and for all $x, z, \xi \in \boldsymbol{R}^n$. Let us suppose that (3.16) holds for $\alpha = 0$ and $|\beta + \gamma| < H$; we can get the same estimate for $\alpha = 0$ and $|\beta + \gamma| = H$ arguing as in the proof of Lemma 3.2; details are omitted for sake of brevity. The case $\alpha \ne 0$ can be proved by similar arguments. Finally, using induction we can also prove that there exist $A_2, B_2 > 0$ such that

$$\left| D_\xi^\gamma D_x^\beta \left[D_y^\alpha \left[(\partial_\eta^\sigma \Phi_\varepsilon) \left(\frac{\eta - \xi + \tilde{\nabla}_x \varphi(x, y, \xi) - \nabla_x \varphi(x, \xi)}{\langle \xi \rangle} \right) \right] \right]_{|y = x+z} \right|$$
$$\le A_2 B_2^{|\alpha| + |\beta| + |\gamma| + |\sigma|}(|\gamma| + |\sigma|)!^\mu (|\alpha| + |\beta|)!^\nu \langle \xi \rangle^{-|\gamma| - |\sigma|} \langle x \rangle^{-|\alpha| - |\beta|} \tag{3.17}$$

for all $\alpha, \beta, \gamma, \sigma \in N^n$ and for all $x, z, \xi \in \boldsymbol{R}^n$. The estimates (3.14), (3.15), (3.16) and (3.17) give directly (3.13). Now, integrating by parts, it follows that

$$r_2(x, \xi) = \sum_{j \ge 0} \psi_j(x, \xi)(j+1) \sum_{|\alpha| = j+1} (\alpha!)^{-1} \int_0^1 (1-t)^j \sum_{q \ge 0} \frac{\sigma^q}{(q!)^{2\theta}} \iint e^{-i\langle z, \zeta \rangle} \frac{1}{m_{2\theta, \sigma}(z)}$$
$$\cdot (1 - \Delta_\zeta)^q [\langle \zeta \rangle^{-2n} (1 - \Delta_z)^n (D_y^\alpha \partial_\eta^\alpha b)(x, x+z, \xi, \xi + t\zeta)] \, dz đ\zeta.$$

Using (3.13) and arguing as before, we obtain the estimate (3.9) also for r_2. This concludes the proof.

REMARK 4. *With the same notation of Theorem* 3.1, *if* $a \sim \sum_{j\geq 0} a_j$ *in* $FS^{\infty}_{\mu,\nu,\theta}(\boldsymbol{R}^{2n})$, *then*

$$q(x,\xi) \sim \sum_{h\geq 0}\sum_{r=0}^{h}\sum_{|\alpha|=h-r}(\alpha!)^{-1}D_y^{\alpha}((\partial_\xi^{\alpha}p)(x,\tilde{\nabla}_x\varphi(x,y,\xi))a_r(y,\xi))_{|_{y=x}}$$

REMARK 5. *In the following, for every* $m=(m_1,m_2)\in\boldsymbol{R}^2$, *we shall denote by* $\Gamma^m_{1,1}(\boldsymbol{R}^{2n})$ *the space of all functions* $a(x,\xi)\in C^{\infty}(\boldsymbol{R}^{2n})$ *such that*

$$\sup_{\alpha,\beta\in\boldsymbol{N}^n}\ \sup_{(x,\xi)\in\boldsymbol{R}^{2n}} C^{-|\alpha|-|\beta|}(\alpha!\beta!)^{-1}\langle\xi\rangle^{-m_1+|\alpha|}\langle x\rangle^{-m_2+|\beta|}|D_\xi^{\alpha}D_x^{\beta}a(x,\xi)|<+\infty$$

for some $C>0$ *independent of* α,β. *We remark that* $\Gamma^m_{1,1}(\boldsymbol{R}^{2n})$ *is a subspace of* $\Gamma^m_{\mu,\nu}(\boldsymbol{R}^{2n})$ *for any choice of* μ,ν *with* $1<\mu\leq\nu$. *Hence, the results of Sections* 2, 3 *hold for pseudodifferential operators with symbol in* $\Gamma^m_{1,1}(\boldsymbol{R}^{2n})$.

4 The Cauchy Problem for SG-hyperbolic Operators with One Characteristic of Constant Multiplicity

Let μ be a real number such that $\mu>1$ and consider the operator

$$P(t,x,D_t,D_x)=(D_t-\lambda(t,x,D_x))^m+\sum_{j=1}^{m}a_j(t,x,D_x)(D_t-\lambda(t,x,D_x))^{m-j}$$

where we assume that for some $T>0$ the operators $\lambda(t,x,D_x), a_j(t,x,D_x)$ satisfy the following conditions:

$$\lambda(t,x,\xi)\ \textit{is real-valued and}\ \lambda\in C^{m-1}([-T,T],\Gamma^e_{1,1}(\boldsymbol{R}^{2n})) \tag{4.1}$$

$$a_j(t,x,\xi)\in C([-T,T],\Gamma^{(pj,qj)}_{\mu,\mu}(\boldsymbol{R}^{2n})),\quad j=1,\dots,m \tag{4.2}$$

for some $p,q\in[0,1[$ such that $p+q<1/(2\mu-1)$. As an application of the calculus for Fourier integral operators developed in the previous sections we want to prove the existence of a solution for the Cauchy problem

$$\begin{cases} P(t,x,D_t,D_x)u=f(t,x) & (t,x)\in[-T,T]\times\boldsymbol{R}^n\\ D_t^k u(s,x)=g_k(x) & k=0,\dots,m-1, x\in\boldsymbol{R}^n\end{cases} \tag{4.3}$$

for some $s\in[-T,T]$, where $f\in C([-T,T],S^{\theta}_{\theta}(\boldsymbol{R}^n))$ and $g_k\in S^{\theta}_{\theta}(\boldsymbol{R}^n)$, $k=0,\dots,m-1$, with $p+q<1/\theta<1/(2\mu-1)$. We want to express the solution by means of a parametrix given by a matrix of Fourier integral operators with symbols in $\Gamma^{\infty}_{\mu,\theta+1-\mu,\theta}(\boldsymbol{R}^{2n})$.

The first step is to determine the phase function φ. As standard, possibly after shrinking T, φ will be determined as solution of the eikonal equation

$$\frac{\partial \varphi}{\partial t} = \lambda(t, x, \nabla_x \varphi(t, x, \eta)) \quad (t, x) \in [-T, T] \times \boldsymbol{R}^n \tag{4.4}$$

with initial condition at $s \in [-T, T]$

$$\varphi_{|_{t=s}} = \langle x, \eta \rangle.$$

Let us consider the Hamilton-Jacobi system

$$\begin{cases} \dot{x}(t) = -\nabla_\xi \lambda(t, x, \xi) & t \in [-T, T] \\ \dot{\xi}(t) = \nabla_x \lambda(t, x, \xi) \end{cases} \tag{4.5}$$

with initial conditions at $s \in [-T, T]$

$$\begin{cases} x_{|_{t=s}} = y \\ \xi_{|_{t=s}} = \eta \end{cases}$$

for some $y, \eta \in \boldsymbol{R}^n$. In view of the assumptions on λ, there exists a unique solution $(x(t, s; y, \eta), \xi(t, s; y, \eta))$ of (4.5). Furthermore, the solution is defined on the whole interval $[-T, T]$ and it satisfies the condition

$$\langle x \rangle \simeq \langle y \rangle, \quad \langle \xi \rangle \simeq \langle \eta \rangle.$$

See [5] and Propositions 4.8 and 4.9 in [6]. We want to prove the following result.

Proposition 4.1. *Under the assumption* (4.1), *the solution* $(x(t, s; y, \eta), \xi(t, s; y, \eta))$ *of* (4.5) *satisfies the following conditions*:

$$x \in C^m([-T, T]^2, \Gamma^{e_2}_{1,1}(\boldsymbol{R}^{2n})),$$
$$\xi \in C^m([-T, T]^2, \Gamma^{e_1}_{1,1}(\boldsymbol{R}^{2n})).$$

Let I be a compact interval of $\boldsymbol{R}$ and denote by I^d the cartesian product of d copies of $I, d \geq 1$.

Lemma 4.2. *Let* $m = (m_1, m_2) \in \boldsymbol{R}^2$, $\varrho \geq 1$ *and* $a \in C^k(I^d, \Gamma^m_{\varrho,\varrho}(\boldsymbol{R}^{2n}))$. *Consider the vectors* $q = (q_1, \ldots, q_n)$, $p = (p_1, \ldots, p_n)$, *where* $q_j \in C^k(I^d, \Gamma^{e_2}_{\varrho,\varrho}(\boldsymbol{R}^{2n}))$ *and* $p_j \in C^k(I^d, \Gamma^{e_1}_{\varrho,\varrho}(\boldsymbol{R}^{2n}))$, $j = 1, \ldots, n$, *are real-valued symbols such that* $\langle q \rangle \simeq \langle x \rangle$ *and* $\langle p \rangle \simeq \langle \xi \rangle$. *Then, for every* $r \in \boldsymbol{N}^d$, $0 \leq |r| \leq k$, *there exist positive constants* A_r, B_r *such that*

$$\sup_{t\in I^d} |D_\xi^\alpha D_x^\beta D_t^r(\partial_\xi^\gamma \partial_x^\delta a)(t, q(t,x,\xi), p(t,x,\xi))|$$
$$\le A_r B_r^{|\alpha|+|\beta|+|\gamma|+|\delta|}(|\alpha|+|\beta|+|\gamma|+|\delta|)!^\varrho \langle\xi\rangle^{m_1-|\alpha|-|\gamma|}\langle x\rangle^{m_2-|\beta|-|\delta|} \tag{4.6}$$

for all $(x,\xi)\in \boldsymbol{R}^{2n}$, $\alpha,\beta,\gamma,\delta\in \boldsymbol{N}^n$. *In particular,* $a(t,q(t,x,\xi),p(t,x,\xi))$ *belongs to* $C^k(I^d,\Gamma^m_{\varrho,\varrho}(\boldsymbol{R}^{2n}))$.

PROOF. To simplify the notations, we will prove (4.6) for $r=0$. The general case does not present further difficulties. We argue by induction on $|\alpha+\beta|$. For $\alpha=\beta=0$, we have

$$|(\partial_\xi^\gamma \partial_x^\delta a)(t,q(t,x,\xi),p(t,x,\xi))| \le AB^{|\gamma|+|\delta|}(|\gamma|+|\delta|)!^\varrho\langle p\rangle^{m_1-|\gamma|}\langle q\rangle^{m_2-|\delta|}$$
$$\le A_0 B_0^{|\gamma|+|\delta|}(|\gamma|+|\delta|)!^\varrho\langle \xi\rangle^{m_1-|\gamma|}\langle x\rangle^{m_2-|\delta|}$$

for all $\gamma,\delta\in \boldsymbol{N}^n$, $t\in I^d$. On the other hand, we can assume that there exist $M,K>0$ such that

$$|D_\xi^\alpha D_x^\beta p_j(t,x,\xi)| \le MK^{|\alpha|+|\beta|}(\alpha!\beta!)^\varrho\langle\xi\rangle^{1-|\alpha|}\langle x\rangle^{-|\beta|}$$
$$|D_\xi^\alpha D_x^\beta q_j(t,x,\xi)| \le MK^{|\alpha|+|\beta|}(\alpha!\beta!)^\varrho\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{1-|\beta|}$$

for all $j=1,\dots,n$. The case $(\alpha,\beta)\neq(0,0)$ can then be treated assuming by induction that for some $h\in \boldsymbol{N}$

$$\sup_{t\in I^d} |D_\xi^\alpha D_x^\beta(\partial_\xi^\gamma \partial_x^\delta a)(t,q(t,x,\xi),p(t,x,\xi))|$$
$$\le A_0 B_0^{|\gamma|+|\delta|}(4\cdot 2^\varrho(n+1)MKB_0)^{|\alpha|+|\beta|}$$
$$\cdot(|\alpha|+|\beta|+|\gamma|+|\delta|)!^\varrho\langle\xi\rangle^{m_1-|\alpha|-|\gamma|}\langle x\rangle^{m_2-|\beta|-|\delta|} \tag{4.7}$$

for all $(x,\xi)\in \boldsymbol{R}^{2n}$, $\gamma,\delta\in \boldsymbol{N}^n$, $|\alpha+\beta|\le h$ and differentiating $|h|+1$ times with respect to x,ξ. The arguments are the same used in Lemma 3.2. □

PROOF OF PROPOSITION 4.1. Let us write the solution of (4.5) in the form

$$\begin{cases} x(t,s;y,\eta) = y + x_0(t,s;y,\eta) \\ \xi(t,s;y,\eta) = \eta + \xi_0(t,s;y,\eta) \end{cases}$$

According to standard arguments, we set

$$\begin{cases} x_{0j}^{(1)}(t,s;y,\eta) = -\int_s^t(\partial_{\xi_j}\lambda)(\tau;y,\eta)\,d\tau & j=1,\dots,n \\ \xi_{0j}^{(1)}(t,s;y,\eta) = \int_s^t(\partial_{x_j}\lambda)(\tau;y,\eta)\,d\tau & j=1,\dots,n \end{cases}$$

and for $k > 1$,

$$\begin{cases} x_{0j}^{(k)}(t,s;y,\eta) = -\int_s^t (\partial_{\xi_j}\lambda)(\tau; y + x_0^{(k-1)}(\tau,s;y,\eta), \eta + \xi_0^{(k-1)}(\tau,s;y,\eta))\,d\tau \\ \xi_{0j}^{(k)}(t,s;y,\eta) = \int_s^t (\partial_{x_j}\lambda)(\tau; y + x_0^{(k-1)}(\tau,s;y,\eta), \eta + \xi_0^{(k-1)}(\tau,s;y,\eta))\,d\tau. \end{cases}$$

We know that the sequence $(x_0^{(k)}, \xi_0^{(k)})$ converges to (x_0, ξ_0) when $k \to +\infty$. Hence, to estimate (x_0, ξ_0) it is sufficient to estimate the terms of the sequence uniformly with respect to k. We claim that there exist positive constants M_0, K such that

$$|D_\eta^\alpha D_y^\beta D_t^r D_s^h x_{0j}^{(k)}(t,s;y,\eta)| \le M_0 K^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!\langle\eta\rangle^{-|\alpha|}\langle y\rangle^{1-|\beta|} \tag{4.8}$$

$$|D_\eta^\alpha D_y^\beta D_t^r D_s^h \xi_{0j}^{(k)}(t,s;y,\eta)| \le M_0 K^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!\langle\eta\rangle^{1-|\alpha|}\langle y\rangle^{-|\beta|} \tag{4.9}$$

for all $j = 1, \dots, n$, $\alpha, \beta \in \boldsymbol{N}^n$, $0 \le h$, $r \le m$ and the inequalities hold for all k in $\boldsymbol{Z}_+$, $y, \eta \in \boldsymbol{R}^n$ and $|t-s| \le T_{\alpha\beta}$, for a suitable constant $T_{\alpha\beta}$ independent of k, y, η and such that $0 < T_{\alpha\beta} < T$. For simplicity, we consider only the case $h = r = 0$ and prove only (4.8), the proof of (4.9) being similar. For $k = 1$, by (4.1), we have

$$|D_\eta^\alpha D_y^\beta x_{0j}^{(1)}(t,s;y,\eta)| \le |t-s| A_\lambda B_\lambda^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!\langle\eta\rangle^{-|\alpha|}\langle y\rangle^{1-|\beta|}$$

for some constants A_λ, B_λ depending on λ. This implies (4.8) for $|t-s| \le 1$. For $k > 1$, suppose that (4.8) and (4.9) are true for $k-1$ and prove them true for k. Applying the arguments in the proof of Lemma 4.2, we can prove that there exist $A_{1\lambda}, B_{1\lambda} > 0$ independent of k such that

$$\begin{aligned} &|D_\eta^\alpha D_y^\beta (\partial_{\xi_j}\lambda)(\tau; y + x_0^{(k-1)}, \eta + \xi_0^{(k-1)})| \\ &\qquad \le A_{1\lambda}(8(n+1)M_0 K B_{1\lambda})^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!\langle\eta\rangle^{-|\alpha|}\langle y\rangle^{1-|\beta|}. \end{aligned}$$

This implies (4.8) for $|t-s| \le T_{\alpha\beta} = M_0 A_{1\lambda}^{-1}(8(n+1)M_0 B_{1\lambda})^{-|\alpha+\beta|}$. From (4.8) and (4.9), we obtain an estimate of the solution of (4.5) for all $t \in [s - T_{\alpha\beta}, s + T_{\alpha\beta}]$. We want to extend this estimate on the whole interval $[-T, T]$. To this end, let $t^* \in [-T, T]$ and consider the shifted system

$$\begin{cases} \dot{X}_j = -(\partial_{\xi_j}\lambda)(t; X, \Xi) \\ \dot{\Xi}_j = (\partial_{x_j}\lambda)(t; X, \Xi) \qquad j = 1, \dots, n \\ X(s) = y^*, \Xi(s) = \eta^* \end{cases} \tag{4.10}$$

where $y^* = x(t^*, s; y, \eta)$, $\eta^* = \xi(t^*, s; y, \eta)$. Repeating the previous arguments for (4.10), we obtain the estimates (4.8) and (4.9) for $X_0 = X - y^*$, $\Xi_0 = \Xi - \eta^*$ with the same constants $M_0, K, T_{\alpha\beta}$. Thus, choosing $t^* \in]s - T_{\alpha\beta}, s + T_{\alpha\beta}[$, in a finite number of steps and choosing suitably larger constants M_0, K, we obtain the estimates (4.8) and (4.9) on the whole interval $[-T, T]$. Analogously, we obtain the estimates for $0 < r \leq m$. These give the continuity of the derivatives with respect to t up to the order $m-1$. The continuity of the m-th derivative directly follows from the equations (4.5). Finally, by (4.1) and by the regularity with respect to t, it follows that the solution $(x(t,s;y,\eta), \xi(t,s;y,\eta))$ of (4.5) is continuously differentiable up to the order m also with respect to s. $\square$

By Proposition 4.1 and Lemma 4.2, we are able to prove that the solution φ of (4.4) is in $C^1([-T', T']^2, \Gamma^e_{1,1}(\boldsymbol{R}^{2n}))$ for some positive $T' < T$. In fact, we observe that the solution $x(t,s;y,\eta)$ of (4.5) is invertible with respect to y for $t \in]s - \tilde{T}, s + \tilde{T}[$, for some $\tilde{T} < T$, because

$$\frac{\partial x}{\partial y}_{|_{t=s}} = I.$$

Denoting by $y = \bar{q}(t,s;x,\eta)$ the inverse function, it is easy to prove using the same arguments of the proof of Proposition 4.1, that $\bar{q} \in C^m([-T', T']^2, \Gamma^{e_2}_{1,1}(\boldsymbol{R}^{2n}))$ for $T' < \tilde{T}$. Finally, we set as standard in the Hamilton-Jacobi theory

$$\begin{aligned}\psi(t,s;y,\eta) = \langle y,\eta\rangle + \int_s^t &[\lambda(\tau; x(\tau,s;y,\eta), \xi(\tau,s;y,\eta)) \\ &- \langle(\nabla_\xi \lambda)(\tau; x(\tau,s;y,\eta), \xi(\tau,s;y,\eta)), \xi(\tau,s;y,\eta)\rangle]\, d\tau\end{aligned}$$

and

$$\varphi(t,s;x,\eta) = \psi(t,s;\bar{q}(t,s;x,\eta),\eta).$$

By the assumptions on λ and by Proposition 4.1, it turns out that ψ belongs to $C^m([-T', T']^2, \Gamma^e_{1,1}(\boldsymbol{R}^{2n}))$, then by Lemma 4.2, the same holds for φ. Moreover, φ is a solution of (4.4), cf. [17] for classical pseudodifferential symbols, [6], [8] for C^∞ SG-symbols.

In order to find a parametrix for the problem (4.3), it is convenient to reduce the equation to a first order system. To this end, we denote by $A_{(p,q)}$ the pseudodifferential operator with symbol $\langle x\rangle^q \langle\xi\rangle^p$ in $\Gamma^{(p,q)}_{1,1}(\boldsymbol{R}^{2n})$ and by $A^{-1}_{(p,q)}$ its inverse given by the product of operators $\langle D\rangle^{-p}\langle x\rangle^{-q}$ with symbol in $\Gamma^{(-p,-q)}_{1,1}(\boldsymbol{R}^{2n})$ and set

$$\begin{cases} u_0 = A^{m-1}_{(p,q)} u \\ u_1 = A^{m-2}_{(p,q)}(D_t - \lambda)u \\ \dots\dots\dots\dots\dots\dots \\ u_{m-2} = A_{(p,q)}(D_t - \lambda)^{m-2} u \\ u_{m-1} = (D_t - \lambda)^{m-1} u \end{cases} \tag{4.11}$$

Using these formulas, it follows that

$$D_t u_i(t,x) = \lambda(t,x,D_x)u_i(t,x) + A_{(p,q)}u_{i+1}(t,x) + [A_{(p,q)}^{m-1-i}, \lambda](A_{(p,q)}^{-1})^{m-1-i}u_i(t,x)$$

for $i = 0, \dots, m-2$ and

$$D_t u_{m-1}(t,x) = \lambda(t,x,D_x)u_{m-1}(t,x) - \sum_{j=1}^{m} a_j(t,x,D_x)(A_{(p,q)}^{-1})^{j-1}u_{m-j} + f(t,x).$$

Denoting by $\sigma^{m-1-i} = [A_{(p,q)}^{m-1-i}, \lambda](A_{(p,q)}^{-1})^{m-1-i}$, $i = 0, \dots, m-2$, the equation (4.3) is equivalent to the system

$$D_t U = \Lambda U + QU + NU + F$$

where

$$U = \begin{pmatrix} u_0 \\ \vdots \\ \vdots \\ u_{m-1} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda & 0 & \cdots & \cdots & 0 \\ 0 & \lambda & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \lambda \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & A_{(p,q)} & 0 & \cdots & 0 \\ 0 & 0 & A_{(p,q)} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 0 & A_{(p,q)} \\ -a_m(A_{(p,q)}^{-1})^{m-1} & -a_{m-1}(A_{(p,q)}^{-1})^{m-2} & -a_{m-2}(A_{(p,q)}^{-1})^{m-3} & \cdots & -a_1 \end{pmatrix}$$

$$N = \begin{pmatrix} \sigma^{m-1} & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \sigma^{m-2} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \sigma & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ f \end{pmatrix}$$

with initial conditions

$$U(s,x) = U_0(x) = \begin{pmatrix} h_0(x) \\ \vdots \\ \vdots \\ h_{m-1}(x) \end{pmatrix}$$

where $h_k(x) \in S_\theta^\theta(\mathbf{R}^n)$, $k = 0, \dots, m-1$, depend on the functions g_k according to (4.11). We observe that all the coefficients of Q have order (p, q) and those of N have order $(0,0)$. Then we can write the system in the form

$$\begin{cases} L(t, x, D_t, D_x)U = F \\ U(s, x) = U_0(x) \end{cases} \tag{4.12}$$

denoting by

$$L(t, x, D_t, D_x) = D_t - \Lambda - M$$

where M is a matrix of operators with symbols $m_{jk} \in C([-T', T'], \Gamma_{\mu,\mu}^{(p,q)}(\mathbf{R}^{2n}))$, $j, k = 1, \dots, m$. We start by considering the homogeneous system and look for an $m \times m$ matrix $E(t, s) = \{E_{jk}(t, s)\}_{j,k=1}^m$ of Fourier integral operators with phase function $\varphi(t, s; x, \eta)$ given by the solution of (4.4) and satisfying

$$\begin{cases} L(t, x, D_t, D_x)E(t, s) = R(t, s) \quad t, s \in [-T', T'] \\ E(s, s) = iI \end{cases} \tag{4.13}$$

where $R(t, s)$ is an $m \times m$ matrix of Fourier integral operators with kernel in $C([-T', T']^2, S_\theta^\theta(\mathbf{R}^{2n}))$ and I is the $m \times m$ diagonal matrix $\operatorname{diag}[\mathbf{1}, \dots, \mathbf{1}]$, where $\mathbf{1}$ is the identity operator on $S_\theta^\theta(\mathbf{R}^n)$. As standard we determine the symbols $e_{jk}(t, s; x, \eta)$, $j, k = 1, \dots, m$, starting from their asymptotic expansions $\sum_{h \ge 0} e_{jk}^{(h)}$. Applying Remark 4 and recalling that $\partial_t \varphi = \lambda(t, x, \nabla_x \varphi)$, it follows that $(LE)_{jk}$ has symbol $b_{jk}(t, s; x, \eta) \sim \sum_{h \ge 0} b_{jk}^{(h)}(t, s; x, \eta)$ where

$$\begin{aligned} b_{jk}^{(0)}(t, s; x, \eta) &= D_t e_{jk}^{(0)}(t, s; x, \eta) - \sum_{r=1}^n (\partial_{\xi_r} \lambda)(t; x, \nabla_x \varphi) D_{x_r} e_{jk}^{(0)}(t, s; x, \eta) \\ &\quad - q(t, s; x, \eta) e_{jk}^{(0)}(t, s; x, \eta) - \sum_{l=1}^m m_{jl}(t; x, \nabla_x \varphi) e_{lk}^{(0)}(t, s; x, \eta) \end{aligned} \tag{4.14}$$

and, for $h \ge 1$,

$$\begin{aligned} &b_{jk}^{(h)}(t, s; x, \eta) \\ &\quad = D_t e_{jk}^{(h)}(t, s; x, \eta) - \sum_{r=1}^n (\partial_{\xi_r} \lambda)(t; x, \nabla_x \varphi) D_{x_r} e_{jk}^{(h)}(t, s; x, \eta) \\ &\quad - q(t, s; x, \eta) e_{jk}^{(h)}(t, s; x, \eta) - \sum_{l=1}^m m_{jl}(t; x, \nabla_x \varphi) e_{lk}^{(h)}(t, s; x, \eta) \end{aligned}$$

$$-\sum_{r=0}^{h-1}\sum_{|\alpha|=h-r+1}(\alpha!)^{-1}D_z^{\alpha}((\partial_{\xi}^{\alpha}\lambda)(t;x,\tilde{\nabla}_x\varphi(t,s;x,z,\eta))e_{jk}^{(r)}(t,s;z,\eta))_{|_{z=x}}$$

$$-\sum_{l=1}^{m}\sum_{r=0}^{h-1}\sum_{|\alpha|=h-r}(\alpha!)^{-1}D_z^{\alpha}((\partial_{\xi}^{\alpha}m_{jl})(t;x,\tilde{\nabla}_x\varphi(t,s;x,z,\eta))e_{lk}^{(r)}(t,s;z,\eta))_{|_{z=x}} \quad (4.15)$$

for all $x,\eta\in \boldsymbol{R}^n$ and for all $t,s\in[-T',T']$, where

$$q(t,s;x,\eta)=-\frac{i}{2}\sum_{r,s=1}^{n}(\partial_{\xi_r\xi_s}^2\lambda)(t;x,\nabla_x\varphi)(\partial_{x_rx_s}^2\varphi)(t,s;x,\eta).$$

By Leibniz formula, it follows that

$$\begin{aligned}b_{jk}^{(h)}(t,s;x,\eta)&=D_te_{jk}^{(h)}(t,s;x,\eta)-\sum_{r=1}^{n}(\partial_{\xi_r}\lambda)(t;x,\nabla_x\varphi)D_{x_r}e_{jk}^{(h)}(t,s;x,\eta)\\&\quad-q(t,s;x,\eta)e_{jk}^{(h)}(t,s;x,\eta)-\sum_{l=1}^{m}m_{jl}(t;x,\nabla_x\varphi)e_{lk}^{(h)}(t,s;x,\eta)\\&\quad-\sum_{l=1}^{m}\sum_{r=0}^{h-1}\sum_{|\beta|\le h-r+1}p_{\beta,h-r,j,l}(t,s;x,\eta)D_x^{\beta}e_{lk}^{(r)}(t,s;x,\eta)\end{aligned}\quad (4.16)$$

where

$$\begin{aligned}p_{\beta,h-r,j,l}(t,s;x,\eta)&=\delta_l^j\sum_{\substack{\alpha\ge\beta\\|\alpha|=h-r+1}}\frac{1}{\beta!(\alpha-\beta)!}D_z^{\alpha-\beta}((\partial_{\xi}^{\alpha}\lambda)(t;x,\tilde{\nabla}_x\varphi(t,s;x,z,\eta)))_{|_{z=x}}\\&\quad+\sum_{\substack{\alpha\ge\beta\\|\alpha|=h-r}}\frac{1}{\beta!(\alpha-\beta)!}D_z^{\alpha-\beta}((\partial_{\xi}^{\alpha}m_{jl})(t;x,\tilde{\nabla}_x\varphi(t,s;x,z,\eta)))_{|_{z=x}}.\end{aligned}$$

It is easy to prove that

$$p_{\beta,h-r,j,l}\in C([-T',T']^2,\Gamma_{\mu,\mu}^{(p-h+r,q-h+r+|\beta|)}(\boldsymbol{R}^{2n})) \quad (4.17)$$

and that

$$\begin{aligned}&\sup_{(t,s)\in[-T',T']^2}|D_\eta^{\gamma}D_x^{\delta}p_{\beta,h-r,j,l}(t,s;x,\eta)|\\&\quad\le AB^{|\gamma|+|\delta|+h-r+1}(|\gamma|+h-r)!^{\mu}\\&\qquad\cdot(|\delta|+h-r-|\beta|+1)!^{\mu}(h-r)!^{-1}\langle\eta\rangle^{p-h+r-|\gamma|}\langle x\rangle^{q-h+r+|\beta|-|\delta|}\end{aligned}\quad (4.18)$$

for some positive constants A, B. Furthermore, for $|\beta| = h - r + 1$,

$$p_{\beta,h-r,j,l}(t,s;x,\eta) = \delta_l^j \frac{1}{\beta!}(\partial_\xi^\beta \lambda)(t;x,\nabla_x\varphi(t,s;x,\eta)).$$

Then, in this case, $p_{\beta,h-r,j,l} \in C([-T',T']^2, \Gamma_{1,1}^{(-h+r,1)}(\boldsymbol{R}^{2n}))$ and

$$\sup_{(t,s)\in[-T',T']^2} |D_\eta^\gamma D_x^\delta p_{\beta,h-r,j,l}(t,s;x,\eta)| \\ \le AB^{|\gamma|+|\delta|+h-r+1}|\gamma|!|\delta|!\langle\eta\rangle^{-h+r-|\gamma|}\langle x\rangle^{1-|\delta|}. \tag{4.19}$$

If $x = x(t,s;y,\eta)$ is the solution of (4.5), then (4.14) and (4.16) give

$$\tilde{b}_{jk}^{(0)}(t,s;y,\eta) = (D_t - \tilde{q}(t,s;y,\eta))\tilde{e}_{jk}^{(0)}(t,s;y,\eta) \\ - \sum_{l=1}^m \tilde{m}_{jl}(t,s;y,\eta)\tilde{e}_{lk}^{(0)}(t,s;y,\eta) \tag{4.20}$$

and

$$\tilde{b}_{jk}^{(h)}(t,s;y,\eta) \\ = (D_t - \tilde{q}(t,s;y,\eta))\tilde{e}_{jk}^{(h)}(t,s;y,\eta) - \sum_{l=1}^m \tilde{m}_{jl}(t,s;y,\eta)\tilde{e}_{lk}^{(h)}(t,s;y,\eta) \\ - \sum_{l=1}^m \sum_{r=0}^{h-1} \sum_{|\beta|\le h-r+1} \tilde{p}_{\beta,h-r,j,l}(t,s;y,\eta) D_z^\beta \tilde{e}_{lk}^{(r)}(t,s;\bar{q}(t,s;z,\eta),\eta)|_{z=x(t,s;y,\eta)} \tag{4.21}$$

where

$$\tilde{b}_{jk}^{(h)}(t,s;y,\eta) = b_{jk}^{(h)}(t,s;x(t,s;y,\eta),\eta)$$
$$\tilde{e}_{jk}^{(h)}(t,s;y,\eta) = e_{jk}^{(h)}(t,s;x(t,s;y,\eta),\eta)$$
$$\tilde{m}_{jl}(t,s;y,\eta) = m_{jl}(t;x(t,s;y,\eta),\xi(t,s;y,\eta))$$
$$\tilde{p}_{\beta,h-r,j,l}(t,s;y,\eta) = p_{\beta,h-r,j,l}(t,s;x(t,s;y,\eta),\eta)$$
$$\tilde{q}(t,s;y,\eta) = q(t,s;x(t,s;y,\eta),\eta)$$

for $j,k,l = 1,\dots,m$, $h \in \boldsymbol{N}$. The functions $e_{jk}^{(h)}$ satisfy (4.13), if $\tilde{e}_{jk}^{(h)}$ are solutions of the equations

$$\tilde{b}_{jk}^{(h)}(t,s;y,\eta) = 0 \quad \forall j,\ k = 1,\dots,m,\ h \ge 0 \tag{4.22}$$

with initial conditions

$$\tilde{e}_{jk}^{(h)}(s,s;y,\eta) = i\delta_0^h\delta_k^j$$

for $t, s \in [-T', T']$, $y, \eta \in \boldsymbol{R}^n$. Let us define

$$\tilde{f}_{jk}^{(h)}(t,s;y,\eta) = \tilde{e}_{jk}^{(h)}(t,s;y,\eta) \cdot \exp\left[-i\int_s^t \tilde{q}(\tau,s;y,\eta)\,d\tau\right], \quad h \geq 0. \tag{4.23}$$

It turns out that $\tilde{e}_{jk}^{(h)}$ are solutions of (4.22) if the functions $\tilde{f}_{jk}^{(h)}$ are solutions of

$$D_t\tilde{f}_{jk}^{(0)}(t,s;y,\eta) - \sum_{l=1}^{m}\tilde{m}_{jl}(t,s;y,\eta)\tilde{f}_{lk}^{(0)}(t,s;y,\eta) = 0 \tag{4.24}$$

and

$$D_t\tilde{f}_{jk}^{(h)}(t,s;y,\eta) - \sum_{l=1}^{m}\tilde{m}_{jl}(t,s;y,\eta)\tilde{f}_{lk}^{(h)}(t,s;y,\eta) = g_{hjk}(t,s;y,\eta) \tag{4.25}$$

with initial conditions

$$\tilde{f}_{jk}^{(h)}(s,s;y,\eta) = i\delta_0^h\delta_j^k, \quad h \geq 0,\ j,k = 1,\dots,m$$

where

$$g_{hjk}(t,s;y,\eta) = \sum_{l=1}^{m}\sum_{r=0}^{h-1}\sum_{|\beta|\leq h-r+1}\tilde{d}_{\beta,h-r,j,l}(t,s;y,\eta)D_y^\beta\tilde{f}_{lk}^{(r)}(t,s;y,\eta) \tag{4.26}$$

for some $\tilde{d}_{\beta,h-r,j,l}$ which satisfy (4.18) and (4.19) for some constants $A \geq 1$, $B \geq 1$, in view of the fact that $\tilde{q}(t,s;y,\eta)$ is in $\Gamma_{1,1}^{(0,0)}(\boldsymbol{R}^{2n})$ for all $t,s \in [-T',T']$.

Lemma 4.3. *The functions $\tilde{f}_{jk}^{(0)}$, solutions of (4.24), satisfy the following condition: there exist $A_o \geq 1$, $c > 0$ such that*

$$|D_\eta^\alpha D_y^\beta \tilde{f}_{jk}^{(0)}(t,s;y,\eta)| \leq A_o^{|\alpha|+|\beta|}(\alpha!\beta!)^\mu\langle\eta\rangle^{-|\alpha|}\langle y\rangle^{-|\beta|}$$
$$\cdot \exp[c\langle\eta\rangle^p\langle y\rangle^q|t-s|]\sum_{i=0}^{|\alpha+\beta|}\langle\eta\rangle^{pi}\langle y\rangle^{qi}\frac{|t-s|^i}{i!} \tag{4.27}$$

$$|D_\eta^\alpha D_y^\beta D_t\tilde{f}_{jk}^{(0)}(t,s;y,\eta)| \leq A_o^{|\alpha|+|\beta|}(\alpha!\beta!)^\mu\langle\eta\rangle^{-|\alpha|}\langle y\rangle^{-|\beta|}$$
$$\cdot \exp[c\langle\eta\rangle^p\langle y\rangle^q|t-s|]\sum_{i=1}^{|\alpha+\beta|+1}\langle\eta\rangle^{pi}\langle y\rangle^{qi}\frac{|t-s|^{i-1}}{(i-1)!} \tag{4.28}$$

for all $t,s \in [-T',T']$, $(y,\eta) \in \boldsymbol{R}^{2n}$.

PROOF. For $\alpha = \beta = 0$, (4.27) follows directly from well known estimates for the solutions of the Cauchy problem for ordinary differential equations. See Lemma 4.1 in [11]. Assume that (4.27) is true for $|\alpha + \beta| \le N$ and let $r \in \{1, \dots, n\}$. Then, $D_{y_r}\tilde{f}_{jk}^{(0)}$ are the solutions of

$$\begin{cases} D_t D_{y_r}\tilde{f}_{jk}^{(0)} - \sum_{l=1}^{m} \tilde{m}_{jl} D_{y_r}\tilde{f}_{lk}^{(0)} = \sum_{l=1}^{m} D_{y_r}\tilde{m}_{jl} \cdot \tilde{f}_{lk}^{(0)} \\ D_{y_r}\tilde{f}_{jk}^{(0)}(s, s; y, \eta) = 0 \end{cases}$$

for $j, k = 1, \dots, m$. Denote by $\tilde{f}_{jk}^{(0)}(t, s, \tau; y, \eta)$ the solution of (4.24) such that

$$\tilde{f}_{jk}^{(0)}(\tau, s, \tau; y, \eta) = i\delta_j^k.$$

We observe that $\tilde{f}_{jk}^{(0)}(t, s, \tau; y, \eta)$ satisfies (4.27) for $|\alpha + \beta| \le N$, replacing $|t - s|$ by $|t - \tau|$ in the right-hand side, cf. [4]. Then

$$D_{y_r}\tilde{f}_{jk}^{(0)}(t, s; y, \eta) = \int_s^t \sum_{i,l=1}^{m} \tilde{f}_{ji}^{(0)}(t, s, \tau; y, \eta) D_{y_r}\tilde{m}_{il}(\tau, s; y, \eta)\tilde{f}_{lk}^{(0)}(\tau, s; y, \eta)\, d\tau$$

which we can estimate inductively obtaining (4.27). Similarly, we argue on $D_{\eta_r}\tilde{f}_{jk}^{(0)}$. The estimate (4.28) can be obtained directly from (4.27) and (4.24). □

To find an estimate for the functions $\tilde{f}_{jk}^{(h)}$, $h \ge 1$ we observe that

$$\tilde{f}_{jk}^{(h)}(t, s; y, \eta) = \sum_{\ell=1}^{m} \int_s^t \tilde{f}_{j\ell}^{(0)}(t, s, \tau; y, \eta) g_{h\ell k}(\tau, s; y, \eta)\, d\tau, \quad h \ge 1. \tag{4.29}$$

LEMMA 4.4. *For $h \ge 1$, the functions $\tilde{f}_{jk}^{(h)}(t, s; y, \eta)$, solutions of (4.25), satisfy the following condition: for every $p' \in [p, 1/\theta[$, $q' \in [q, 1/\theta[$ with $(2\mu - \theta)/\theta \le p' + q' < 1/\theta$*

$$\begin{aligned} |D_\eta^\alpha D_y^\beta \tilde{f}_{jk}^{(h)}(t, s; y, \eta)| \le{}& A(4 \cdot 2^{\mu+n} m^2 A_o AB)^{|\alpha|+|\beta|+4h}(|\alpha| + |\beta| + 2h)!^\mu \\ &\cdot (h!)^{-\theta(p'+q')}\langle \eta \rangle^{-|\alpha|-h}\langle y \rangle^{-|\beta|-h} \exp[(c+1)\langle \eta \rangle^p \langle y \rangle^q |t - s|] \\ &\cdot \sum_{i=0}^{|\alpha+\beta|+3h} \langle \eta \rangle^{p'i}\langle y \rangle^{q'i} \frac{|t - s|^i}{i!} \end{aligned} \tag{4.30}$$

for all $(y, \eta) \in \mathbf{R}^{2n}$ with $\langle \eta \rangle \ge h^\theta$ and $\langle y \rangle \ge h^\theta$ and for all $\alpha, \beta \in \mathbf{N}^n$, t, s in $[-T', T']$, where A, B are the constants appearing in (4.18) *and* (4.19) *and A_o, c are* *the constants appearing in* (4.27) *and* (4.28). *Moreover, there exists* $A_1 > 0$ *such that*

$$|D_\eta^\alpha D_y^\beta D_t \tilde{f}_{jk}^{(h)}(t,s;y,\eta)| \le A_1^{|\alpha|+|\beta|+4h+1}(|\alpha|+|\beta|+2h)!^\mu (h!)^{-\theta(p'+q')}$$
$$\cdot \langle\eta\rangle^{-|\alpha|-h}\langle y\rangle^{-|\beta|-h}\exp[(c+1)\langle\eta\rangle^p\langle y\rangle^q|t-s|]$$
$$\cdot \sum_{i=1}^{|\alpha+\beta|+3h+1}\langle\eta\rangle^{p'i}\langle y\rangle^{q'i}\frac{|t-s|^{i-1}}{(i-1)!} \tag{4.31}$$

for all $(y,\eta)\in \boldsymbol{R}^{2n}$ *with* $\langle\eta\rangle \ge h^\theta$ *and* $\langle y\rangle \ge h^\theta$ *and for all* $\alpha,\beta\in \boldsymbol{N}^n$, t,s *in* $[-T',T']$.

Proof. In order to prove (4.30), we argue by induction on $h \ge 1$. For $h=1$, by (4.18), (4.26) and (4.27), it follows that

$$|D_\eta^\gamma D_y^\delta g_{1\ell k}(\tau,s;y,\eta)|$$
$$\le \sum_{l=1}^m \sum_{|\beta|\le 2}\sum_{\gamma'\le\gamma}\binom{\gamma}{\gamma'}\sum_{\delta'\le\delta}\binom{\delta}{\delta'}|D_\eta^{\gamma'}D_y^{\delta'}\tilde{d}_{\beta,1,\ell,l}(\tau,s;y,\eta)|$$
$$\cdot |D_\eta^{\gamma-\gamma'}D_y^{\delta-\delta'+\beta}\tilde{f}_{lk}^{(0)}(\tau,s;y,\eta)|$$
$$\le Am(A_oB)^{|\gamma|+|\delta|+2}\langle\eta\rangle^{-|\gamma|-1}\langle y\rangle^{-|\delta|-1}\exp[c\langle\eta\rangle^p\langle y\rangle^q|\tau-s|]$$
$$\cdot\sum_{i=0}^{|\gamma+\delta|+2}\langle\eta\rangle^{p(i+1)}\langle y\rangle^{q(i+1)}\frac{|\tau-s|^i}{i!}$$
$$\cdot\sum_{|\beta|\le 2}\sum_{\gamma'\le\gamma}\binom{\gamma}{\gamma'}\sum_{\delta'\le\delta}\binom{\delta}{\delta'}(|\gamma-\gamma'|+|\delta-\delta'|+|\beta|)!^\mu(|\gamma'|+|\delta'|-|\beta|+3)!^\mu.$$

By (2.5), we have $(|\gamma'|+|\delta'|-|\beta|+3)!^\mu \le 2^{\mu(|\gamma'|+|\delta'|-|\beta|+3)}(|\gamma'|+|\delta'|-|\beta|+2)!^\mu$, from which we deduce that

$$\sum_{|\beta|\le 2}\sum_{\gamma'\le\gamma}\binom{\gamma}{\gamma'}\sum_{\delta'\le\delta}\binom{\delta}{\delta'}(|\gamma-\gamma'|+|\delta-\delta'|+|\beta|)!^\mu(|\gamma'|+|\delta'|-|\beta|+3)!^\mu$$
$$\le 2^{3\mu}(2\cdot 2^\mu)^{|\gamma|+|\delta|}(|\gamma|+|\delta|+2)!^\mu\sum_{|\beta|\le 2}2^{-\mu|\beta|}\le 2^{3\mu+n}(2\cdot 2^\mu)^{|\gamma|+|\delta|}(|\gamma|+|\delta|+2)!^\mu.$$

Hence

$$
\begin{aligned}
|D_\eta^\gamma D_y^\delta g_{1\ell k}(\tau,s;y,\eta)| \le{}& 2^{3\mu+n}Am(A_oB)^2(2\cdot 2^\mu A_oB)^{|\gamma|+|\delta|}(|\gamma|+|\delta|+2)!^\mu \\
&\cdot \langle\eta\rangle^{-|\gamma|-1}\langle y\rangle^{-|\delta|-1}\exp[c\langle\eta\rangle^p\langle y\rangle^q|\tau-s|] \\
&\cdot \sum_{i=0}^{|\gamma|+|\delta|+2}\langle\eta\rangle^{p(i+1)}\langle y\rangle^{q(i+1)}\frac{|\tau-s|^i}{i!}. \qquad (4.32)
\end{aligned}
$$

Furthermore, by (4.27), it follows that

$$
\begin{aligned}
&|D_\eta^\gamma D_y^\delta \tilde f_{j\ell}^{(0)}(t,s,\tau;y,\eta)| \\
&\le A_o^{|\gamma|+|\delta|}(\gamma!\delta!)^\mu\langle\eta\rangle^{-|\gamma|}\langle y\rangle^{-|\delta|}\exp[(c+1)\langle\eta\rangle^p\langle y\rangle^q|t-\tau|]. \qquad (4.33)
\end{aligned}
$$

for all $j,\ell=1,\dots,m$. By (4.29), combining (4.32) and (4.33), we deduce that

$$
\begin{aligned}
|D_\eta^\alpha D_y^\beta \tilde f_{jk}^{(1)}(t,s;y,\eta)| \le{}& A(4\cdot 2^{n+\mu}m^2A_oAB)^{|\alpha|+|\beta|+4}(|\alpha|+|\beta|+2)!^\mu \\
&\cdot \langle\eta\rangle^{-|\alpha|-1}\langle y\rangle^{-|\beta|-1}\exp[(c+1)\langle\eta\rangle^p\langle y\rangle^q|t-s|] \\
&\cdot \sum_{i=0}^{|\alpha|+|\beta|+2}\frac{\langle\eta\rangle^{p(i+1)}\langle y\rangle^{q(i+1)}}{i!}\int_s^t|\tau-s|^i\,d\tau
\end{aligned}
$$

for all $(y,\eta)\in \boldsymbol{R}^{2n}$, $t,s\in[-T',T']$, $\alpha,\beta\in \boldsymbol{N}^n$, which gives in particular (4.30) for $h=1$. For $h>1$, by (4.26), we can decompose

$$
D_\eta^\gamma D_y^\delta g_{h\ell k}(\tau,s;y,\eta)=\sum_{r=0}^{h-1}(\lambda_{\gamma,\delta,h,r,\ell,k}(\tau,s;y,\eta)+\sigma_{\gamma,\delta,h,r,\ell,k}(\tau,s;y,\eta))
$$

where

$$
\begin{aligned}
\lambda_{\gamma,\delta,h,r,\ell,k}(\tau,s;y,\eta)={}&\sum_{l=1}^m\sum_{|\beta|\le h-r}\sum_{\gamma'\le\gamma}\binom{\gamma}{\gamma'}\sum_{\delta'\le\delta}\binom{\delta}{\delta'}D_\eta^{\gamma'}D_y^{\delta'}\tilde d_{\beta,h-r,\ell,l}(\tau,s;y,\eta) \\
&\cdot D_\eta^{\gamma-\gamma'}D_y^{\delta-\delta'+\beta}\tilde f_{lk}^{(r)}(\tau,s;y,\eta) \qquad (4.34)
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma_{\gamma,\delta,h,r,\ell,k}(\tau,s;y,\eta)={}&\sum_{l=1}^m\sum_{|\beta|=h-r+1}\sum_{\gamma'\le\gamma}\binom{\gamma}{\gamma'}\sum_{\delta'\le\delta}\binom{\delta}{\delta'}D_\eta^{\gamma'}D_y^{\delta'}\tilde d_{\beta,h-r,\ell,l}(\tau,s;y,\eta) \\
&\cdot D_\eta^{\gamma-\gamma'}D_y^{\delta-\delta'+\beta}\tilde f_{lk}^{(r)}(\tau,s;y,\eta). \qquad (4.35)
\end{aligned}
$$

Assuming (4.30) true for $r=1,\dots,h-1$ and observing that $|\beta|+3r\le 3h-1$, by (4.18) and (4.27), it follows that

$$
\begin{aligned}
&|\lambda_{\gamma,\delta,h,r,\ell,k}(\tau,s;y,\eta)| \\
&\quad \le A^2 m(4\cdot 2^{\mu+n}m^2A_oAB)^{|\gamma|+|\delta|+4r}(h-r)!^{-1}(r!)^{-\theta(p'+q')}\langle\eta\rangle^{-|\gamma|-h}\langle y\rangle^{-|\delta|-h} \\
&\qquad \cdot \exp[(c+1)\langle\eta\rangle^p\langle y\rangle^q|\tau-s|]\sum_{i=0}^{|\gamma|+|\delta|+3h-1}\langle\eta\rangle^{p'(i+1)}\langle y\rangle^{q'(i+1)}\frac{|\tau-s|^i}{i!} \\
&\qquad \cdot B^{h-r+1}\sum_{|\beta|\le h-r}(4\cdot 2^{\mu+n}m^2A_oAB)^{|\beta|}\sum_{\gamma'\le\gamma}\binom{\gamma}{\gamma'}\frac{1}{2^{(\mu+1)|\gamma'|}}(|\gamma'|+h-r)!^{\mu} \\
&\qquad \cdot\sum_{\delta'\le\delta}\binom{\delta}{\delta'}\frac{1}{2^{(\mu+1)|\delta'|}}(|\delta'|+h-r-|\beta|+1)!^{\mu}(|\gamma-\gamma'|+|\delta-\delta'|+|\beta|+2r)!^{\mu}.
\end{aligned}
$$

By (2.5), we have $(|\delta'|+h-r-|\beta|+1)!^{\mu}\le 2^{\mu(|\delta'|+h-r-|\beta|+1)}(|\delta'|+h-r-|\beta|)!^{\mu}$. Then, applying Lemma 5.1 in [14], it turns out that

$$
\begin{aligned}
&\sum_{\gamma'\le\gamma}\binom{\gamma}{\gamma'}\frac{1}{2^{(\mu+1)|\gamma'|}}(|\gamma'|+h-r)!^{\mu}\sum_{\delta'\le\delta}\binom{\delta}{\delta'}\frac{1}{2^{(\mu+1)|\delta'|}}(|\delta'|+h-r-|\beta|+1)!^{\mu} \\
&\qquad \cdot(|\gamma-\gamma'|+|\delta-\delta'|+|\beta|+2r)!^{\mu} \\
&\quad \le 2\cdot 2^{\mu(h-r-|\beta|+1)}\sum_{\gamma'\le\gamma}\binom{\gamma}{\gamma'}\frac{1}{2^{(\mu+1)|\gamma'|}}(|\gamma'|+h-r)!^{\mu}(|\gamma-\gamma'|+|\delta|+h+r)!^{\mu} \\
&\quad \le 4\cdot 2^{\mu(h-r+1-|\beta|)}(|\gamma|+|\delta|+2h)!^{\mu}\binom{|\delta|+2h}{h-r}^{-1}.
\end{aligned}
$$

Now, observing that

$$
\binom{|\delta|+2h}{h-r}^{-1}(h-r)!^{-1}(r!)^{-\theta(p'+q')}\le(h!)^{-\theta(p'+q')}
$$

and that

$$
\sum_{|\beta|\le h-r}2^{-\mu|\beta|}(4\cdot 2^{\mu+n}m^2A_oAB)^{|\beta|}\le 2^n(4\cdot 2^{\mu+n}m^2A_oAB)^{h-r},
$$

we obtain

$$
\begin{aligned}
&\sum_{r=0}^{h-1}|\lambda_{\gamma,\delta,h,r,\ell,k}(\tau,s;y,\eta)| \\
&\qquad \le A^2m(4\cdot 2^{\mu+n}m^2A_oAB)^{|\gamma|+|\delta|+4h}(|\gamma|+|\delta|+2h)!^{\mu}
\end{aligned}
$$

$$\cdot (h!)^{-\theta(p'+q')}\langle\eta\rangle^{-|\gamma|-h}\langle y\rangle^{-|\delta|-h}\exp[(c+1)\langle\eta\rangle^{p}\langle y\rangle^{q}|\tau-s|]$$

$$\cdot\sum_{i=0}^{|\gamma|+|\delta|+3h-1}\langle\eta\rangle^{p'(i+1)}\langle y\rangle^{q'(i+1)}\frac{|\tau-s|^{i}}{i!}$$

$$\cdot 4\cdot 2^{n}\sum_{r=0}^{h-1}(2^{\mu}B)^{h-r+1}(4\cdot 2^{\mu+n}m^{2}A_{o}AB)^{3(r-h)}$$

from which, in particular, we deduce that

$$\sum_{r=0}^{h-1}|\lambda_{\gamma,\delta,h,r,\ell,k}(\tau,s;y,\eta)|$$
$$\leq\frac{A}{8m}(4\cdot 2^{\mu+n}m^{2}A_{o}AB)^{|\gamma|+|\delta|+4h}(|\gamma|+|\delta|+2h)!^{\mu}$$
$$\cdot (h!)^{-\theta(p'+q')}\langle\eta\rangle^{-|\gamma|-h}\langle y\rangle^{-|\delta|-h}\exp[(c+1)\langle\eta\rangle^{p}\langle y\rangle^{q}|\tau-s|]$$
$$\cdot\sum_{i=0}^{|\gamma|+|\delta|+3h-1}\langle\eta\rangle^{p'(i+1)}\langle y\rangle^{q'(i+1)}\frac{|\tau-s|^{i}}{i!}. \tag{4.36}$$

for all $(y,\eta)\in\boldsymbol{R}^{2n}$ with $\langle\eta\rangle\geq h^{\theta}$ and $\langle y\rangle\geq h^{\theta}$ and for all τ,s in $[-T',T']$, $\gamma,\delta\in\boldsymbol{N}^{n}$. Let us now estimate (4.35). By applying (4.19), (4.27) and the inductive hypothesis, we obtain

$$\sum_{r=0}^{h-1}|\sigma_{\gamma,\delta,h,r,\ell,k}(\tau,s;y,\eta)|$$
$$\leq A^{2}m\langle\eta\rangle^{-|\gamma|-h}\langle y\rangle^{-|\delta|-h}\exp[(c+1)\langle\eta\rangle^{p}\langle y\rangle^{q}|\tau-s|]$$
$$\cdot\sum_{i=0}^{|\gamma|+|\delta|+3h-1}\langle\eta\rangle^{p'i}\langle y\rangle^{q'i}\frac{|\tau-s|^{i}}{i!}\cdot 4\sum_{r=0}^{h-1}(4\cdot 2^{\mu+n}m^{2}A_{o}AB)^{|\gamma|+|\delta|+h+3r+1}$$
$$\cdot B^{h-r+1}(r!)^{-\theta(p'+q')}(|\gamma|+|\delta|+h+r+1)!^{\mu}\sum_{|\beta|=h-r+1}1.$$

In view of the fact that $\langle\eta\rangle\geq h^{\theta}$ and $\langle y\rangle\geq h^{\theta}$, it follows that

$$(r!)^{-\theta(p'+q')}(|\gamma|+|\delta|+h+r+1)!^{\mu}\leq(|\gamma|+|\delta|+2h)!^{\mu}(h-1)!^{-\theta(p'+q')}$$
$$\leq(|\gamma|+|\delta|+2h)!^{\mu}(h!)^{-\theta(p'+q')}\langle\eta\rangle^{p'}\langle y\rangle^{q'}.$$

Moreover, assuming A_o sufficiently large in (4.27) and (4.28) and arguing as before, it is easy to prove that

$$4\sum_{r=0}^{h-1}(4\cdot 2^{\mu+n}m^2A_oAB)^{|\gamma|+|\delta|+h+3r+1}$$

$$\cdot B^{h-r+1}\sum_{|\beta|=h-r+1}1\leq \frac{1}{8Am^2}(4\cdot 2^{\mu+n}m^2A_oAB)^{|\gamma|+|\delta|+4h}.$$

Then, it turns out that

$$\sum_{r=0}^{h-1}|\sigma_{\gamma,\delta,h,r,\ell,k}(\tau,s;y,\eta)|$$

$$\leq \frac{A}{8m}(4\cdot 2^{\mu+n}m^2A_oAB)^{|\gamma|+|\delta|+4h}$$

$$\cdot(|\gamma|+|\delta|+2h)!^{\mu}(h!)^{-\theta(p'+q')}\langle\eta\rangle^{-|\gamma|-h}\langle y\rangle^{-|\delta|-h}\exp[(c+1)\langle\eta\rangle^p\langle y\rangle^q|\tau-s|]$$

$$\cdot\sum_{i=0}^{|\gamma|+|\delta|+3h-1}\langle\eta\rangle^{p'(i+1)}\langle y\rangle^{q'(i+1)}\frac{|\tau-s|^i}{i!}. \tag{4.37}$$

Arguing as in the case $h=1$, by (4.33), (4.36) and (4.37) we obtain (4.30). The estimate (4.31) directly follows from (4.25) and (4.30). □

Lemma 4.5. *For $h\geq 1$, the functions $\tilde{f}_{jk}^{(h)}(t,s;y,\eta)$, solutions of* (4.25), *satisfy the following estimates*:

$$|D_\eta^\alpha D_y^\beta\tilde{f}_{jk}^{(h)}(t,s;y,\eta)|\leq A(4\cdot 2^{\mu+n}m^2A_oAB)^{|\alpha|+|\beta|+4h}(|\alpha|+|\beta|+2h)!^{\mu}(h!)^{-\theta(p'+q')}$$

$$\cdot\langle\eta\rangle^{-|\alpha|-h}\langle y\rangle^{-|\beta|-h}\exp[(c+1)\langle\eta\rangle^p\langle y\rangle^q|t-s|]$$

$$\cdot\sum_{i=0}^{|\alpha+\beta|+3h}\langle\eta\rangle^{(p'+q')i}\frac{|t-s|^i}{i!} \tag{4.38}$$

and

$$|D_\eta^\alpha D_y^\beta D_t\tilde{f}_{jk}^{(h)}(t,s;y,\eta)|\leq A_1^{|\alpha|+|\beta|+4h+1}(|\alpha|+|\beta|+2h)!^{\mu}(h!)^{-\theta(p'+q')}$$

$$\cdot\langle\eta\rangle^{-|\alpha|-h}\langle y\rangle^{-|\beta|-h}\exp[(c+1)\langle\eta\rangle^p\langle y\rangle^q|t-s|]$$

$$\cdot\sum_{i=1}^{|\alpha+\beta|+3h+1}\langle\eta\rangle^{(p'+q')i}\frac{|t-s|^{i-1}}{(i-1)!} \tag{4.39}$$

for all $(y,\eta) \in \boldsymbol{R}^{2n}$ *with* $\langle \eta \rangle \geq h^{\theta}$ *and* $\langle y \rangle < h^{\theta}$ *and for all* t,s *in* $[-T',T']$, α,β *in* $\boldsymbol{N}^n$, *where* p',q' *are the same of Lemma* 4.4.

PROOF. As in Lemma 4.4, it is sufficient to prove (4.38) by induction on h, observing that, for $\langle \eta \rangle \geq h^{\theta}$ and $\langle y \rangle < h^{\theta}$, we have $\langle \eta \rangle^{p'} \langle y \rangle^{q'} \leq \langle \eta \rangle^{p'+q'}$. Then, arguing as in Lemma 4.4 and applying (4.30) and the inductive assumption (4.38) for $r = 1, \dots, h-1$, we deduce that

$$\begin{aligned}
&\sum_{r=0}^{h-1} |\lambda_{\gamma,\delta,h,r,\ell,k}(\tau,s;y,\eta)| \\
&\quad \leq \frac{A}{8m}(4 \cdot 2^{\mu+n} m^2 A_o AB)^{|\gamma|+|\delta|+4h}(|\gamma|+|\delta|+2h)!^{\mu} \\
&\qquad \cdot (h!)^{-\theta(p'+q')} \langle \eta \rangle^{-|\gamma|-h} \langle y \rangle^{-|\delta|-h} \exp[(c+1)\langle \eta \rangle^{p} \langle y \rangle^{q} |\tau - s|] \\
&\qquad \cdot \sum_{i=0}^{|\gamma|+|\delta|+3h-1} \langle \eta \rangle^{(p'+q')(i+1)} \frac{|\tau - s|^{i}}{i!}.
\end{aligned} \tag{4.40}$$

Furthermore, observing that

$$\begin{aligned}
(r!)^{-\theta(p'+q')}(|\gamma|+|\delta|+h+r+1)!^{\mu} &\leq (|\gamma|+|\delta|+2h)!^{\mu}(h-1)!^{-\theta(p'+q')} \\
&\leq (|\gamma|+|\delta|+2h)!^{\mu}(h!)^{-\theta(p'+q')}\langle \eta \rangle^{p'+q'}
\end{aligned}$$

for $\langle \eta \rangle \geq h^{\theta}$ and arguing as in Lemma 4.4, we have

$$\begin{aligned}
&\sum_{r=0}^{h-1} |\sigma_{\gamma,\delta,h,r,\ell,k}(\tau,s;y,\eta)| \\
&\leq \frac{A}{8m}(4 \cdot 2^{\mu+n} m^2 A_o AB)^{|\gamma|+|\delta|+4h} \\
&\quad \cdot (|\gamma|+|\delta|+2h)!^{\mu}(h!)^{-\theta(p'+q')} \langle \eta \rangle^{-|\gamma|-h} \langle y \rangle^{-|\delta|-h} \exp[(c+1)\langle \eta \rangle^{p} \langle y \rangle^{q} |\tau - s|] \\
&\quad \cdot \sum_{i=0}^{|\gamma|+|\delta|+3h-1} \langle \eta \rangle^{(p'+q')(i+1)} \frac{|\tau - s|^{i}}{i!}.
\end{aligned} \tag{4.41}$$

The estimates (4.33), (4.40) and (4.41) imply (4.38). □

REMARK 6. *By the same arguments of Lemma* 4.5, *it is easy to prove that the following estimates hold*:

$$|D_\eta^\alpha D_y^\beta \tilde{f}_{jk}^{(h)}(t,s;y,\eta)| \le A(4\cdot 2^{\mu+n}m^2A_oAB)^{|\alpha|+|\beta|+4h}(|\alpha|+|\beta|+2h)!^\mu(h!)^{-\theta(p'+q')}$$
$$\cdot \langle\eta\rangle^{-|\alpha|-h}\langle y\rangle^{-|\beta|-h}\exp[(c+1)\langle\eta\rangle^p\langle y\rangle^q|t-s|]$$
$$\cdot\sum_{i=0}^{|\alpha+\beta|+3h}\langle y\rangle^{(p'+q')i}\frac{|t-s|^i}{i!} \tag{4.42}$$

and

$$|D_\eta^\alpha D_y^\beta D_t\tilde{f}_{jk}^{(h)}(t,s;y,\eta)| \le A_1^{|\alpha|+|\beta|+4h+1}(|\alpha|+|\beta|+2h)!^\mu(h!)^{-\theta(p'+q')}$$
$$\cdot \langle\eta\rangle^{-|\alpha|-h}\langle y\rangle^{-|\beta|-h}\exp[(c+1)\langle\eta\rangle^p\langle y\rangle^q|t-s|]$$
$$\cdot\sum_{i=1}^{|\alpha+\beta|+3h+1}\langle y\rangle^{(p'+q')i}\frac{|t-s|^{i-1}}{(i-1)!} \tag{4.43}$$

for all $(y,\eta)\in\boldsymbol{R}^{2n}$ *with* $\langle y\rangle\ge h^\theta$ *and* $\langle\eta\rangle<h^\theta$ *and for all* t,s *in* $[-T',T']$, $\alpha,\beta\in\boldsymbol{N}^n$, *where* p',q' *are the same of Lemmas* 4.4 *and* 4.5.

Theorem 4.6. *There exists an* $m\times m$ *matrix* $E(t,s)=\{E_{jk}\}_{j,k=1}^m$ *of Fourier integral operators with phase function* $\varphi(t,s;x,\eta)$ *as before and symbols* $e_{jk}(t,s;x,\eta)$ *in* $C^1([-T',T']^2,\Gamma^\infty_{\mu,\theta+1-\mu,\theta}(\boldsymbol{R}^{2n}))$ *for all* $j,k=1,\dots,m$, *which satisfies* (4.13).

Proof. We observe that the condition $(2\mu-\theta)/\theta\le p'+q'<1/\theta$ in Lemmas 4.4 and 4.5 and Remark 6 implies that $2\mu-\theta(p'+q')\le\theta$. Then, it follows that there exists $C>0$ such that for every $\varepsilon>0$ and $r=0,1$

$$\sup_{h\in N}\sup_{\alpha,\beta\in N^n}\sup_{(x,\eta)\in Q^e_{h^\theta}}\sup_{t,s\in[-T',T']} C^{-|\alpha|-|\beta|-2h}(\alpha!\beta!)^{-\mu}(h!)^{-\theta}$$
$$\cdot\langle\eta\rangle^{|\alpha|+h}\langle x\rangle^{|\beta|+h}\exp[-\varepsilon(|x|^{1/\theta}+|\eta|^{1/\theta})]|D_\eta^\alpha D_x^\beta D_t^r\tilde{f}_{jk}^{(h)}(t,s;x,\eta)|<+\infty. \tag{4.44}$$

Furthermore, by (4.23), recalling that $\tilde{q}$ has order $(0,0)$ and arguing as in Lemma 4.2, it is easy to show that, eventually enlarging the constants, the estimates (4.44) hold also for the functions $\tilde{e}_{jk}^{(h)}$ and consequently for $e_{jk}^{(h)}$. By (4.44) and by the condition $\theta>2\mu-1$, it follows in particular that $\sum_{h\ge0}e_{jk}^{(h)}$ and $\sum_{h\ge0}D_te_{jk}^{(h)}$ are bounded in $FS^\infty_{\mu,\theta+1-\mu,\theta}(\boldsymbol{R}^{2n})$ uniformly with respect to $t,s\in[-T',T']$. Starting from $\sum_{h\ge0}e_{jk}^{(h)}$, we can argue as in Theorem 2.14 and find a sequence of functions $\varphi_h(x,\eta)\in C^\infty(\boldsymbol{R}^{2n})$ depending on a parameter R such that $e_{jk}=\sum_{h\ge0}\varphi_he_{jk}^{(h)}$ is in $\Gamma^\infty_{\mu,\theta+1-\mu,\theta}(\boldsymbol{R}^{2n})$, $j,k=1,\dots,m$ if R is sufficiently large. Moreover, the functions $\varphi_he_{jk}^{(h)}$ belong to $C([-T',T']^2,\Gamma^\infty_{\mu,\theta+1-\mu,\theta}(\boldsymbol{R}^{2n}))$ for ever $h\ge0$, $j,k=1,\dots,$

m. Then, it is easy to prove that also $e_{jk} \in C([-T', T']^2, \Gamma^{\infty}_{\mu,\theta+1-\mu,\theta}(\boldsymbol{R}^{2n}))$ for all $j, k = 1, \dots, m$. The same argument can be used to prove that also $D_t e_{jk}$ is in $C([-T', T']^2, \Gamma^{\infty}_{\mu,\theta+1-\mu,\theta}(\boldsymbol{R}^{2n}))$ for all $j, k = 1, \dots, m$. Finally, we can conclude that $E(t, s)$ satisfies (4.13). □

We conclude giving a theorem of existence of a solution for the Cauchy problem (4.3).

THEOREM 4.7. *Let $f \in C([-T, T], S^{\theta}_{\theta}(\boldsymbol{R}^n))$ and $g_k \in S^{\theta}_{\theta}(\boldsymbol{R}^n)$, $k = 0, \dots, m-1$. Under the assumptions* (4.1), (4.2), *there exists a positive $T' < T$ such that, for every $s \in [-T', T']$, the problem*

$$\begin{cases} P(t, x, D_t, D_x)u = f(t, x) & (t, x) \in [-T', T'] \times \boldsymbol{R}^n \\ D_t^k u(s, x) = g_k(x) & k = 0, \dots, m-1,\ x \in \boldsymbol{R}^n \end{cases}$$

admits a solution $u \in C^m([-T', T'], S^{\theta}_{\theta}(\boldsymbol{R}^n))$. An analogous result holds when we replace $S^{\theta}_{\theta}(\boldsymbol{R}^n)$ with $S^{\theta\prime}_{\theta}(\boldsymbol{R}^n)$.

PROOF. Obviously, it is sufficient to prove that the problem (4.12) has a solution $U(t, x) \in C^1([-T', T'], (S^{\theta}_{\theta}(\boldsymbol{R}^n))^m)$, where we denote by $(S^{\theta}_{\theta}(\boldsymbol{R}^n))^m$ the cartesian product of m copies of $S^{\theta}_{\theta}(\boldsymbol{R}^n)$. We start by considering the case $U_0(x) = 0$. We look for a solution of the form

$$U(t, x) = \int_s^t E(t, \tau)[F(\tau, x) + H(\tau, x)]\, d\tau \tag{4.45}$$

for a suitable $H(\tau, x) \in C([-T', T'], (S^{\theta}_{\theta}(\boldsymbol{R}^n))^m)$. To this end, denote, for any $G \in C([-T', T'], (S^{\theta}_{\theta}(\boldsymbol{R}^n))^m)$

$$\mathscr{R}G(t, x) = \int_s^t R(t, \tau)G(\tau, x)\, d\tau$$

where $R(t, \tau) = \{R_{jk}\}_{j,k=1}^m$ is the matrix of θ-regularizing operators with kernel in $C([-T', T']^2, S^{\theta}_{\theta}(\boldsymbol{R}^{2n}))$ appearing in (4.13). The function $U(t, x)$ defined by (4.45) is a solution if and only if the function $H(t, x)$ satisfies the relation

$$H(t, x) + \mathscr{R}H(t, x) + \mathscr{R}F(t, x) = 0 \tag{4.46}$$

for every $(t, x) \in [-T', T'] \times \boldsymbol{R}^n$. To prove (4.46), we can limit ourselves to show that the Neumann series $\sum_{v=1}^{\infty}(-1)^v \mathscr{R}^v F(t, \cdot)$ converges uniformly with respect to $t \in [-T', T']$ in $(S^{\theta}_{\theta}(\boldsymbol{R}^n))^m$. Let us prove the convergence of the single components.

With reference to the norms in (1.1), there exist positive integers A, B such that

$$\|(\mathscr{R}F)_j(t,\cdot)\|_{A,B,n} \le K\int_s^t \left(\int_{\boldsymbol{R}^n} |f(\tau,y)|\, dy\right) d\tau$$

where $K = \max_{j,k=1,\dots,m} \sup_{t,s\in[-T',T']} \|K_{R_{jk}}(t,s,\cdot,\cdot)\|_{A,B,2n}$, with $K_{R_{jk}}$ kernel of the operator R_{jk}, $j,k = 1,\dots m$. In particular, it follows that

$$\|(\mathscr{R}F)_j(t,\cdot)\|_{A,B,n} \le K' \cdot \sup_{t\in[-T',T']} \|f(t,\cdot)\|_{A,B,n} \cdot |t-s|$$

for some positive $K' > K$. Suppose that

$$\|(\mathscr{R}^{\nu}F)_j(t,\cdot)\|_{A,B,n} \le (K'm)^{\nu} \sup_{t\in[-T',T']} \|f(t,\cdot)\|_{A,B,n} \frac{|t-s|^{\nu}}{\nu!} \tag{4.47}$$

and prove (4.47) true for $\nu+1$. We have

$$\begin{aligned}
&\|(\mathscr{R}^{\nu+1}F)_j(t,\cdot)\|_{A,B,n} \\
&= \left\| \sum_{l=1}^{m} \int_s^t R_{jl}(t,\tau)(\mathscr{R}^{\nu}F)_l(t,\cdot) \right\|_{A,B,n} \\
&\le K\sum_{l=1}^{m} \int_s^t \|(\mathscr{R}^{\nu}F)_l(\tau,\cdot)\|_{A,B,n}\, d\tau \le (mK')^{\nu+1} \sup_{t\in[-T',T']} \|f(t,\cdot)\|_{A,B,n} \int_s^t \frac{|\tau-s|^{\nu}}{\nu!}\, d\tau \\
&\le (mK')^{\nu+1} \sup_{t\in[-T',T']} \|f(t,\cdot)\|_{A,B,n} \frac{|t-s|^{\nu+1}}{(\nu+1)!}.
\end{aligned}$$

Hence, the convergence of the series is proved and we have a solution for the problem (4.12) when $U_0 = 0$. Arguing as before, we can prove that there exists $\tilde{H} \in C([-T',T'], (S_\theta^\theta(\boldsymbol{R}^n))^m)$ such that

$$\tilde{U}(t,x) = U_0(x) - \int_s^t E(t,\tau)[LU_0(\tau,x) + \tilde{H}(\tau,x)]\, d\tau$$

is a solution of the problem

$$\begin{cases} L\tilde{U}(t,x) = 0 & (t,x) \in [-T',T'] \times \boldsymbol{R}^n \\ \tilde{U}(s,x) = U_0(x). \end{cases} \tag{4.48}$$

Combining the two solutions, we obtain a solution for (4.12). □

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