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On $(n-1)$-dimensional projective spaces contained in the Grassmann variety $\mathrm{Gr}(n, 1)$

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§0. Introduction

In this paper, we understand by a variety a projective variety which is defined over a fixed algebraically closed field k of characteristic p (which can be zero).

Our main purpose of the present paper is to classify the type of subvarieties of $\mathrm{Gr}(n, 1)$ which are biregular to projective spaces of dimension $n-1$.[1)]

As examples of such varieties we know followings.

$$X^0_{n,1} = \left\{\begin{pmatrix} 1, & 0, & 0,\dots, & 0 \\ 0, & x_0, & x_1,\dots, & x_{n-1} \end{pmatrix} \in \mathrm{Gr}(n, 1) \mid (x_0, x_1,\dots, x_{n-1}) \in \mathbf{P}^{n-1}\right\}^{2)}.$$

$$X^1_{n,1} = \left\{\begin{pmatrix} x_0, & x_1,\dots, & x_{n-1}, & 0 \\ 0, & x_0,\dots, & x_{n-2}, & x_{n-1} \end{pmatrix} \in \mathrm{Gr}(n, 1) \mid (x_0, x_1,\dots, x_{n-1}) \in \mathbf{P}^{n-1}\right\}.$$

$$\check{X}^0_{3,1} = \phi_3(X^0_{3,1})$$

$$\check{X}^1_{3,1} = \phi_3(X^1_{3,1})$$

where ϕ_n: $\mathrm{Gr}(n, 1) \to \mathrm{Gr}(n, n-2)$ is the dual biregular morphism.

1) In general $\mathrm{Gr}(n, d)$ denotes the Grassman variety which paramerizes d-dimensional linear subspace of n-dimensional projective space $\mathbf{P}^n$.

2) By $\begin{pmatrix} 1, & 0, & 0,\dots, & 0 \\ 0, & x_0, & x_1,\dots, & x_{n-1} \end{pmatrix}$ we denote the point of $\mathrm{Gr}(n, 1)$ which represent the line which passes two points $(1, 0, 0,\dots, 0)$ and $(0, x_0, x_1,\dots, x_{n-1})$ of $\mathbf{P}^n$.

$X_q(S)=\{x\in \mathrm{Gr}(4, 1)|$ the line which is represented by x is contained in $S\}$,[3] where S is a non-singular quadric hypersurface of $\mathbf{P}^4$ and $\mathrm{char}\, k=p\neq 2$.

The main Theorems are the following theorems.

Theorem 5.1. *Let X be a subvariety of* $\mathrm{Gr}(n, 1)$ *which is biregular to* $\mathbf{P}^{n-1}$. *Then,*

(i) *if $n=3$, then X is projectively equivalent*[4] *to some one of* $X^0_{3,1}, X^1_{3,1}, \check{X}^0_{3,1}$, *and* $\check{X}^1_{3,1}$.

(ii) *if $n\geqq 5$, then X is projectively equivalent to $X^0_{n,1}$ or to $X^1_{n,1}$.*

Theorem 6.2. *Assume that the characteristic of k is not equal to 2. Let X be a subvariety of* $\mathrm{Gr}(4, 1)$ *which is biregular to* $\mathbf{P}^3$. *Then, X is projectively equivalent to some one of $X^0_{4,1}, X^1_{4,1}$ and $X_q(S)$, where S is a fixed non-singular quadric hypersurface of* $\mathbf{P}^4$.

We shall prove these theorems by numerical method. Let $E(n, 1)$ (resp. $Q(n, 1)$) be the universal subbundle (resp universal quotient bundle) of $\mathrm{Gr}(n, 1)$. Assume that X is a subvariety of $\mathrm{Gr}(n, 1)$ which is biregular to $\mathbf{P}^{n-1}$. Let $E=\check{E}(n, 1)|_X$ and $Q=Q(n, 1)|_X$. And let $c_1(E)=hH$ and $c_2(E)=bH^2$ where H is a hyperplane of $X\approx\mathbf{P}^{n-1}$. Then, we shall prove Theorem 5.1 and Theorem 6.2 by completing the following table.

n	(h, b)	Property of X	Type of X
3	$(1, 0)$	$E\approx \mathcal{O}_X\oplus\mathcal{O}_X(1)$	$X^0_{3,1}$
	$(2, 1)$	$E\approx \mathcal{O}_X(1)\oplus\mathcal{O}_X(1)$	$X^1_{3,1}$
	$(1, 1)$	$Q\approx \mathcal{O}_X\oplus\mathcal{O}_X(1)$	$\check{X}^0_{3,1}$
	$(2, 3)$	$Q\approx \mathcal{O}_X(1)\oplus\mathcal{O}_X(1)$	$\check{X}^1_{3,1}$

3) In §6 we shall prove that $X_q(S)$ is biregular to $\mathbf{P}^3$.

4) Subvarieties X and Y of Gr $(n, 1)$ are said to be projectively equivalent to each other if there exists a biregular map σ from $\mathrm{Gr}(n, 1)$ to $\mathrm{Gr}(n, 1)$ which is induced by an element of $PGL(n, k)$ such that $\sigma(X)=Y$.

4	(1, 0)	$E \approx \mathcal{O}_X \oplus \mathcal{O}_X(1)$	$X^0_{4,1}$
	(2, 1)	$E \approx \mathcal{O}_X(1) \oplus \mathcal{O}_X(1)$	$X^1_{4,1}$
	(2, 2)	(*)	$X_q(S)$
$\geqq 5$	(1, 0)	$E \approx \mathcal{O}_X \oplus \mathcal{O}_X(1)$	$X^0_{n,1}$
	(2, 1)	$E \approx \mathcal{O}_X(1) \oplus \mathcal{O}_X(1)$	$X^1_{n,1}$

(*): All the lines which are represented by the points of X are contained in some hypersurface of $\mathbf{P}^4$.

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§1. Notation and preliminary results

As mentioned in the introduction, we understand by a variety a variety defined over an algebraically closed field k of characteristic p. In §1, §2, §3, §4 and §5, p is arbitrary. And in §6, we assume that $p \neq 2$. We consider the Grassmann vairety $\mathrm{Gr}(n, d)$ parametrizing d-dimensional linear subspaces of n-dimensional projective space $\mathbf{P}^n$. If x is a point of $\mathrm{Gr}(n, d)$, we denote by L_x the d-dimensional linear subspace of $\mathbf{P}^n$ which is represented by x.

Let $A_0, A_1, \dots, A_d$ be $d+1$ liner spaces of $\mathbf{P}^n$ such that

$$A_0 \subsetneqq A_1 \subsetneqq \cdots \subsetneqq A_d,$$

and let a_i be the dimension of A_i $(0 \leqq i \leqq d)$. Then the following subvariety of $\mathrm{Gr}(n, d)$

$$\Omega_{a_0, a_1, \dots, a_d}(A_0, A_1, \dots, A_d) = \{x \in \mathrm{Gr}(n, d) | \dim(L_x \cap A_i) \geqq i \quad \text{for all } i\}$$

is called the Schubert variety associated with $A_0, A_1, \dots, A_d$. Two Schubert varieties $\Omega_{a_0, a_1, \dots, a_d}(A_0, A_1, \dots, A_d)$ and $\Omega_{b_0, b_1, \dots, b_d}(B_0, B_1, \dots, B_d)$ are rationaly equivalent to each other if and only if $a_i = b_i$ for all i. The equivalence class containing $\Omega_{a_0, a_1, \dots, a_d}(A_0, A_1, \dots, A_d)$ is denoted by $\Omega_{a_0, a_1, \dots, a_d}$, and is called a Schubert cycle.

Since $\Omega_{0, n-d+1, n-d+2, \dots, n}(A_0, A_1, \dots, A_d)$ depends only on A_0, we also denote it by $\Omega_{0,n-d+1,n-d+2,\dots,n}(A_0)$. Similarly we denote $\Omega_{n-d-1,n-d,\dots,n-1}(A_0, A_1,\dots, A_d)$ by $\Omega_{n-d-1,n-d,\dots,n-1}(A_d)$.

The Schubert cycles $\Omega_{a_0,a_1,\dots,a_d}$ where $a_0, a_1,\dots, a_d$ runs over all integers which satisfy the relation

$$0 \leqq a_0 < a_1 < \cdots < a_d \leqq n \qquad \text{(Schubert condition)}$$

form a free generator of Chow ring $A(\mathrm{Gr}(n, d))$ of $\mathrm{Gr}(n, d)$ as an additive group. The codimension of $\Omega_{a_0,a_1,\dots,a_d}$ is $\sum_{i=0}^{d}(n-d+i-a_i)$. The formula, colled Pieri's formula, show the multiplicative structure of Chow ring $A(\mathrm{Gr}(n, d))$.

$$\Omega_{a_0,a_1,\dots,a_d} \cdot \Omega_{n-d-h,n-d+1,n-d+2,\dots,n} = \sum \Omega_{b_0,b_1,\dots,b_d}$$

where the summation is made over all distinct sets $b_0, b_1,\dots, b_d$ such that

$$0 \leqq b_0 \leqq a_0 < b_1 \leqq a_1 < b_2 \leqq \cdots \leqq a_{d-1} < b_d \leqq a_d \leqq n \quad \text{and}$$

$$\sum_{i=0}^{d} b_i = \sum_{i=0}^{d} a_i - h.$$

In order to describe the structure of $A(\mathrm{Gr}(n, d))$ in simpler way, we set $\omega_{a_0,a_1,\dots,a_d} = \Omega_{n-d-a_0,n-d+1-a_1,\dots,n-a_d}$. Then, Schubert cycles $\{\omega_{a_0,a_1,\dots,a_d}\}$ where $a_0, a_1,\dots, a_d$ run over all integers which satisfy the relation

$$n-d \geqq a_0 \geqq a_1 \geqq \cdots \geqq a_d \geqq 0$$

form a free generators of Chow ring $A(\mathrm{Gr}(n, d))$, and we have the formula

$$\omega_{a_0,a_1\dots,a_d} \cdot \omega_{h,0,\dots,0} = \sum \omega_{b_0,b_1,\dots,b_d}$$

where the summation is made over all distict sets of integers $b_0, b_1,\dots, b_d$ which satisfy the relation

$$n-d \geqq b_0 \geqq a_0 \geqq b_1 \geqq a_1 \geqq b_2 \geqq \cdots \geqq a_{d-1} \geqq b_d \geqq a_d \geqq 0 \quad \text{and}$$

$$\sum_{i=0}^{d} b_i = \sum_{i=0}^{d} a_i + h.$$

The codimension of $\omega_{a_0, a_1, \ldots, a_d}$ is $\sum_{i=0}^{d} a_i$.

Let $d_1, d_2, \ldots, d_s$ be a set of integers with $n \geqq d_1 > d_2 > \cdots > d_s \geqq 0$, then the subvariety $\{(x_1, x_2, \ldots, x_s) \mid \mathbf{P}^n \supset L_{x_1} \subset L_{x_2} \supset \cdots \subset L_{x_s}\}$ of $\mathrm{Gr}(n, d_1) \times \mathrm{Gr}(n, d_2) \times \cdots \times \mathrm{Gr}(n, d_s)$ is called the flag variety of type $(n, d_1, d_2, \ldots, d_s)$ and is denoted by $\mathrm{Dr}(n, d_1, d_2, \ldots, d_s)$.

Lemma 1.1. *For a subvariety* Z *of* $\mathrm{Gr}(n, d)$ *with* $\dim Z \geqq \sum_{i=0}^{d} a_i$, *and for a general point* $(x_d, x_{d-1}, \ldots, x_0)$ *of* $\mathrm{Dr}(n, n-a_d, n-1-a_{d-1}, \ldots, n-d-a_0)$, *we have*

$$\dim (Z \cap \Omega_{n-d-a_0, n-d+1-a_1, \ldots, n-a_d}(L_{x_{n-d-a_0}}, L_{x_{n-d+1-a_0}}, \ldots, L_{x_{n-a_d}}))$$

$$= \dim X - \sum_{i=0}^{d} a_i \quad \text{or} \quad -1 .$$

Proof. Consider the subvariety $X = \{(x, (x_d, x_{d-1}, \ldots, x_0)) \in \mathrm{Gr}(n, d) \times \mathrm{Dr}(n, n-a_d, n-1-a_{d-1}, \ldots, n-d-a_0) \mid \dim (L_x \cap L_{x_i}) \geqq i \text{ for all } i\}$ of $\mathrm{Gr}(n, d) \times \mathrm{Dr}(n, n-a_d, n-1-a_{d-1}, \ldots, n-d-a_0)$. Let $\pi_1; X \to \mathrm{Gr}(n, d)$ and $\pi_2; X \to \mathrm{Dr}(n, n-a_d, n-1-a_{b-1}, \ldots, n-d-a_0)$ be projections. Then,

$$\dim X = \dim \mathrm{Dr}(n, n-a_d, n-1-a_{d-1}, \ldots, n-d-a_0) + \dim \omega_{a_0, a_1, \ldots, a_d}.$$

Since $\pi_1^{-1}(x)$ and $\pi_1^{-1}(y)$ are biregular to each other for any two points x and y of $\mathrm{Gr}(n, d)$, we have

$$\dim \pi_1^{-1}(x) = \dim X - \dim \mathrm{Gr}(n, d).$$

To prove Lemma 1.1, it is enough to show that for some point A of $\mathrm{Dr}(n, n-a_d, n-1-a_{d-1}, \ldots, n-d-a_0)$,

$$\dim (\pi_2^{-1}(A) \cap \pi_1^{-1}(Z)) \leqq \dim Z - \sum_{i=0}^{d} a_i .$$

Assume the contrary. Then for any point A of $\mathrm{Dr}(n, n-a_d, n-1-a_{d-1}, \ldots, n-d-a_0)$

$$\dim(\pi_2^{-1}(A)\cap\pi_1^{-1}(Z))\geqq\dim Z-\sum_{i=0}^{d}a_i+1\,.$$

Hence, we have

$$\dim\pi_1^{-1}(Z)\geqq\dim\mathrm{Dr}(n,\ n-a_d,\ n-1-a_{d-1},\dots,\ n-d-a_0)+\dim Z$$
$$-\sum_{i=0}^{d}a_i+1\,.$$

On the other hand we have

$$\dim\pi_1^{-1}(Z)=\dim Z+\dim X-\dim\mathrm{Gr}(n,\ d)$$
$$=\dim\mathrm{Dr}(n,\ n-a_d,\ n-1-a_{d-1},\dots,\ n-d-a_0)+\dim Z$$
$$-\sum_{i=0}^{d}a_i\,.$$

This is a contradiction. q.e.d.

Corollary 1.2. *The Schubert cycles are numerically non-negative, i.e. the intersection number of Z with $\omega_{a_0,a_1,\dots,a_d}$ is non-negative, for any subavriety Z of dimension $\sum_{i=0}^{d}a_i$ of* Gr(n, d).

Let $E(n, d)$ be the universal subbundle of Gr(n, d) and let $Q(n, d)$ be the universal quotient bundle of Gr(n, d), Then, there exists a canonical exact seqence of vector bundles

$$0\longrightarrow E(n,\ d)\longrightarrow\overset{n+1}{\oplus}\mathcal{O}_{\mathrm{Gr}(n,d)}\longrightarrow Q(n,\ d)\longrightarrow 0\,.$$

Suppose that X is a variety, E is a vector bundle of rank $d+1$ on X and that there exists an exact sequence of vector bundles

$$0\longrightarrow E\longrightarrow\overset{n+1}{\oplus}\mathcal{O}_X\longrightarrow Q\longrightarrow 0\,.$$

Then, there is a canonical morphism

$$f;\ X\longrightarrow\mathrm{Gr}(n,\ d)$$

such that the exact sequence

$$0\longrightarrow E\longrightarrow\overset{n+1}{\oplus}\mathcal{O}_X\longrightarrow Q\longrightarrow 0.$$

is isomorphic to the pull back of

$$0 \longrightarrow E(n, d) \longrightarrow \overset{n+1}{\oplus} \mathcal{O}_{\mathrm{Gr}(n,d)} \longrightarrow Q(n, d) \longrightarrow 0\,.$$

by f.

For a vector bundle E, we denote by $\check{E}$ the dual vector bundle of E. The exact sequence of vector bundle on $\mathrm{Gr}(n, d)$

$$0 \longrightarrow \check{Q}(n, d) \longrightarrow \overset{n+1}{\oplus} \mathcal{O}_{\mathrm{Gr}(n,d)} \longrightarrow \check{E}(n, d) \longrightarrow 0\,.$$

(which is the dual of the exact sequence

$$0 \longrightarrow E(n, d) \longrightarrow \overset{n+1}{\oplus} \mathcal{O}_{\mathrm{Gr}(n,d)} \longrightarrow Q(n, d) \longrightarrow 0)$$

induces a canonical morphism ϕ; $\mathrm{Gr}(n, d) \to \mathrm{Gr}(n, n-d-1)$. It is eaasy to see that ϕ is a biregular map. We denote $\phi(X)$ by $\check{X}$, for any subvariety X of $\mathrm{Gr}(n, d)$. It is easy to see that $(X^{\vee})^{\vee} = X$.

For a vector bundle E on a variety X, we denote by $c_i(E)$ the i-th Chern class of E (which is an element of $A(X)$ of degree i). Then, the following lemma is well known.

Lemma 1.3 *$c_i(\check{E}(n, d)) = \omega_{\underbrace{1, 1, \ldots, 1}_{i}, 0, \ldots, 0}$ if $i \leqq d+1$ and $c_i(E(n, d)) = 0$ if $i > d+1$. (cf. for example* [5]).

The tangent bundle $T_{\mathrm{Gr}(n,d)}$ of $\mathrm{Gr}(n, d)$ is isomorphic to $\check{E}(n, d) \otimes Q(n, d)$. Therefore, we have the following exact sequence

$$0 \longrightarrow \check{E}(n, d) \otimes E(n, d) \longrightarrow \overset{n+1}{\oplus} \check{E}(n, d) \longrightarrow T_{\mathrm{Gr}(n,d)} \longrightarrow 0\,.$$

Let R be a commutative ring with identity and let $R[[t]]$ be the formal power series ring of one variable t with coefficient ring R. For each positive integer i, we define a group homomorphism χ_i; $R[[t]] \to R$ by

$$\chi_i\Big(\sum_{j=0}^{\infty} a_j t^j\Big) = a_i\,.$$

When $c(t) = 1 + c_1 t + c_2 t^2 + \cdots + c_n t^n + \cdots$ is an element of $R[[t]]$, we denote $\chi_i(c(-t)^{-1})$ by $\Phi_i(c(t))$. By definition

$$c(-t)(1+\Phi_1(c(t))t+\Phi_2(c(t))t^2+\cdots+\Phi_n(c(t))t^n+\cdots)=1\,.$$

When $c(t)=1+c_1t+c_2t^2+\cdots+c_nt^n$ is an element of $R[t]$, we also denote $\Phi_i(c(t))$ by $\Phi_i(c_1, c_2,\dots, c_n)$.

Lex X be a non-singular variety of dimension m and let E be a vector bundle on X. The element $c(E)=1+c_1(E)+c_2(E)+\cdots+c_m(E)$ of Chow ring $A(X)$ is called the Chern character of E. For the simplicity, we denote $\Phi_i(c_1(E), c_2(E),\dots, c_m(E))$ by $\Phi_i(c(E))$ and we de note $1+\Phi_1(c(E))+\Phi_2(c(E))+\cdots+\Phi_m(c(E))$ by $\Phi(c(E))$. Then, we have $c(\check{E})\cdot\Phi(c(E))=1$.

Lemma 1.4. *In* $\mathrm{Gr}(n, d)$,

$$c_i(Q(n, d))=\Phi_i(c(\check{E}(n, d))=\omega_{i,0,\dots,0} \qquad (=0 \text{ if } i>n-d)\,.$$

Proof. Assume that the following sequence of vector bundles on a variety X is exact.

$$0\longrightarrow E\longrightarrow \overset{n+1}{\oplus}\mathcal{O}_X\longrightarrow Q\longrightarrow 0\,.$$

Then, we have

$$c(E)c(Q)=1\,.$$

Hence we have

$$c(Q)=\Phi(c(\check{E}))\,.$$

In $\mathrm{Gr}(n, d)$, it is easy to see by the direct calculation that

$$\begin{aligned}0=\omega_{i,0,0,\dots,0}&-\omega_{1,0,\dots,0}\cdot\omega_{i-1,0,\dots,0}\\&+\omega_{1,1,0,\dots,0}\cdot\omega_{i-2,0,\dots,0}-\omega_{1,1,1,0,\dots,0}\cdot\omega_{i-3,0,\dots0},\\&+\cdots+(-1)^{d+1}\omega_{1,1,\dots,1}\cdot\omega_{i-d-1,0,\dots,0}\end{aligned}$$

where $\omega_{j,0,\dots,0}=0$ if $j<0$ or $j>n-d$.

This shows that

$$c_i(Q(n, d)) = \Phi_i(c(\check{E}(n, d))) = \omega_{i,0,\dots,0}. \qquad \text{q.e.d.}$$

§2. Vector bundles generated by their global sections

Proposition 2.1. *Let X be a non-singular variety of dimension m and let E be a vector bundle of arbitrary rank which is generated by its global sections. Then,*

(i) *$c_i(E)$ and $\Phi_i(c(E))$ are numerically non-negative, for all $i=1, 2,\dots, m$.*

(ii) *$c_1(E)c_i(E) - c_{i+1}(E)$ and $c_1(E)\Phi_i(c(E)) - \Phi_{i+1}(c(E))$ are numerically non-negaitve, for all $i=1, 2,\dots, m-1$. In particular if $c_i(E)$ (resp. $\Phi_i(c(E))$) is numerically equivalent to zero, then so is $c_{i+1}(E)$ (resp. $\Phi_{i+1}(c(E))$).*

Proof. Since E is generated by its global sections, we have

$$\overset{n+1}{\oplus} \mathcal{O}_X \longrightarrow E \longrightarrow 0,$$

hence we have

$$0 \longrightarrow \check{E} \longrightarrow \overset{n+1}{\oplus} \mathcal{O}_X \longrightarrow (\text{quotient bundle}) \longrightarrow 0.$$

Then, there exists a canonical morphism $f; X \to \mathrm{Gr}(n, d)$ such that $E = f^*\check{E}(n, d)$ where $d+1$ is the rank of E. Thus we have

$$c_i(E) = f^* c_i(\check{E}(n, d)) = \begin{cases} f^*\omega_{1,1,\dots,1,0,\dots,0} & \text{if } i \leqq d+1 \\ 0 & \text{if } i > d+1. \end{cases}$$

$$\Phi_i(c(E)) = f^*\Phi_i(c(\check{E}(n, d))) = \begin{cases} f^*\omega_{i,0,\dots,0} & \text{if } i \leqq n-d \\ 0 & \text{if } i > n-d. \end{cases}$$

$$c_1(E)c_i(E) - c_{i+1}(E) = \begin{cases} f^*\omega_{2,1,\dots,1,0,\dots,0}, & \text{if } i \leqq d+1 \\ 0 & \text{if } i > d+1. \end{cases}$$

$$c_1(E)\Phi_i(c(E))-\Phi_{i+1}(c(E))=\begin{cases} f^*\omega_{i,1,0,\dots,0} & \text{if } i\leqq n-d \\ 0 & \text{if } i>n-d. \end{cases}$$

Hence (i) and (ii) follow, by virtue of Corollary 1.2 and projection formula.

Proposition 2.2. *Let X be a variety of dimension m and let E be a vector bundle of rank $d+1$. Suppose that E is generated by its global sections, $m\geqq d+1$ and $c_{d+1}(E)=0$. Then*

(i) *There exits a $(m-d)$-dimensional subvariety Y of X such that $E|_Y=\mathcal{O}_Y\oplus E'$ where E' is some vector bundle of rank d on Y.*

(ii) *Suppose $d=1$. Then either E has a trivial line bundle as direct summand or there exists a morphism f from X to a curve C such that $E=f^*E''$ with a suitable vector bundle E'' on C.*

In order to prove Proposition 2.2, we need some preliminaries.

Lemma 2.3. *For a subvaritety X of* $\mathrm{Gr}(n, d)$, *the following three conditions are equivalent to each other.*

(i) $X\cdot\omega_{1,1,\dots,1}=0$.

(ii) *There exists a hyperplane H of $\mathbf{P}^n$ such that H does not contain L_x, for any point x of X.*

(iii) *For a general hyperplane H of $\mathbf{P}^n$, there is no point x of X such that H contains L_x.*

If there exists a non-singular variety $\tilde{X}$ and a morphism f from $\tilde{X}$ onto X, the following conditions are equivalent to these three conditions.

(iv) *$E(n, d)|_X$ has a trivial line bundle as a quotient bundle.*

Proof. (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) are obvious by virtue of Lemma 1.1.
(iv)$\Rightarrow$(i): Since $f^*E(n, d)$ has a trivial line bundle as a aquotient bundle,

$$c_{d+1}(f^*\check{E}(n, d))=f^*c_{d+1}(\check{E}(n, d))=f^*\omega_{1,1,\dots,1}=0$$

Hence we have $X\cdot\omega_{1,1,\dots,1}=0$.

(ii)$\Rightarrow$(iv) is obvious. q.e.d.

Lemma 2.4. *Let X be an m-dimensional subvariety of* Gr(n, d) *which satisfies the conditions* (i)$\sim$(iii) *of Lemma* 2.3, *and assume that $m \geqq d+1$. Then, there exists a $(m-d)$-dimensional subvariety Y of X such that $E(n, d)|_Y$ has a trivial line bundle as a direct summand. If $d=1$, then $E(n, d)|_X$ has a trivial line bnudle as a direct summand.*

Proof. Let x_0 be a point of X, and we consider the following diagram.

$$
\begin{tikzcd}
 & \mathrm{Dr}(n, n-1, d) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathrm{Gr}(n, n-1) & & \mathrm{Gr}(n, d) \supset X \ni x_0
\end{tikzcd}
$$

Set $Z=\pi_1 \circ \pi_2^{-1}(x_0)=\{h \in \mathrm{Gr}(n, n-1) | L_h \supset L_{x_0}\}$ and $W=\pi_1^{-1}(Z) \cap \pi_2^{-1}(X)=\{(h, x) \in \mathrm{Dr}(n, n-1, d) | x \in X, L_h \supset L_x$ and $L_h \supset L_{x_0}\}$
For any point h of Z,

$$\dim \pi_1^{-1}(h) \cap W=\dim(X \cap \omega_{1,1,\dots,1}(L_h)) \geqq \dim X-d.$$

Hence there exists an irreducible component W_0 of W such that

$$\dim W_0 \geqq \dim Z+\dim X-d=\dim X+n-2d-1.$$

Hence, for any point x of $\pi_2^{-1}(W_0)$ we have

$$\dim \pi_2^{-1}(x) \cap W \geqq \dim \pi_2^{-1}(x) \cap W_0=\dim X+n-2d-1-\dim \pi_2(W_0).$$

Since $\pi_2^{-1}(x) \cap W \approx \{h \in \mathrm{Gr}(n, n-1) | L_h \supset L_x$ and $L_h \supset L_{x_0}\}$,

$$\dim \pi_2^{-1}(x) \cap W=n-1-\dim\{\text{linear space spaned by } L_x \text{ and } L_{x_0}\}$$

$$=n-1-(2d-\dim(L_x \cap L_{x_0}))$$

Let $\pi_2(W_0)=Y_0$, then for any point x of Y_0 we have

$$\dim(L_x \cap L_{x_0}) \geqq \dim X - \dim Y_0 . \tag{1}$$

We consider the following diagram.

$$\begin{tikzcd} & \mathrm{Dr}(n, d, 0) \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathrm{Gr}(n, d) \supset X \supset Y_0 & & \mathrm{Gr}(n, 0) \approx \mathbf{P}^n \supset L_{x_0} \end{tikzcd}$$

since

$$\dim(pr_1^{-1}(Y_0) \cap pr_2^{-1}(L_{x_0})) \geqq \dim Y_0 + \dim X - \dim Y_0 = \dim X ,$$

we have for some point P of $pr_2 \circ pr_1^{-1}(Y_0) \cap L_{x_0}$

$$\dim(pr_1^{-1}(Y_0) \cap pr_2^{-1}(P)) \geqq \dim X - \dim L_{x_0} = m - d .$$

This shows that there exists an $(m-d)$-dimensional subvariety Y of X such that for any point x of Y, L_x goes throguh a common point P of $\mathbf{P}^n$. Therefore, $E(n, d)|_Y$ has a trivial line bundle as a direct summand.

If $d=1$, the formula (1) shows that $\dim X = \dim Y_0$, hence $X = Y_0$. This shows that for arbitrary two points x and y of X, L_x and L_y have a common point. This and the condition (ii)) of Lemma 2.3 show that for any point x of X, L_x has a common point. Therefore, $E(n, d)|_X$ has a trivial line bundle as a direct summand. q.e.d.

Proof of Proposition 2.2. Since E is generated by its global sections, we have the following exact sequence.

$$0 \longrightarrow \check{E} \longrightarrow \overset{n+1}{\oplus} \mathcal{O}_X \longrightarrow (\text{quotient bundle}) \longrightarrow 0 .$$

Hence there exists a canonical morphism $f: X \to \mathrm{Gr}(n, d)$ such that $\check{E} = f^* \check{E}(n, d)$. Let $m' = \dim f(X)$. Since $c_{d+1}(E) = 0$, we see that $f(X)$ satisfies the conditions (i)$\sim$(iii) of Lemma 2.3.

If $m' \leqq d$, the assertion is trivial.

Assume that $m' \geqq d+1$. By virtue of Lemma 2.4, there exists an an $(m'-d)$-diemnsional subvariety Y' of $f(X)$, such that $E(n, d)|_{Y'}$ has a trivial line bundle as a direct summand. Since $\dim f^{-1}(Y') \geqq m-d$, there exists an $(m-d)$-dimensional subvariety of X such that $E|_Y$ has a trivial line bundle as a direct summand.

Assume now that $d=1$ and $m' \geqq 2$. By virtue of Lemma 2.4, $E(n, 1)|_{f(X)}$ has a trivial line bundle as a direct summand. This shows that E has a trivial line bundle as direct sumand. q.e.d.

Corollary 2.5. *Let X be an m-dimensional non-singular variety and let E be an ample vector bundle of rank r.* Assume *that E is generated by its global sections. Then, $c^I(E)$ is numerically positive if $|I|$ is less than $m+1$ and $r+1$, where $I=(i_1, i_2, \dots, i_r)$ is a set of non-negative integers,*

$$c^I(E)=c_1(E)^{i_1}\cdot c_2(E)^{i_2}\cdot\cdots\cdot c_r(E)^{i_r} \quad \text{and} \quad |I|=i_1+2i_2+\cdots+ri_r.$$

(Sumihiro [5])

Proof. Since E is generated by its global sections, there exists an exact sequence

$$0 \longrightarrow \check{E} \longrightarrow \overset{n+1}{\oplus}\mathcal{O}_X \longrightarrow (\text{quotient bundle}) \longrightarrow 0.$$

This exact sequence define a morphism $f; X \to \mathrm{Gr}(n, r-1)$, such that $E=f^*\check{E}(n, r-1)$. Let r' be a positive integer such that $r' \leqq \min\{r, m\}$. Suppose that $c_{r'}(E)$ is not numerically positive. Since $c_{r'}(E)$ is numerically non-negative (by virtue of Proposition 2.1), there exists r'-dimensional subvariety Z of X, such that $Z\cdot c_{r'}(E)=0$. Since

$$Z\cdot c_{r'}(E)=Zf^*\omega_{\underbrace{1,1,\dots,1}_{r'},0,\dots,0},$$

we have

$$f(Z)\cdot\omega_{\underbrace{1,1,\dots,1}_{r'},0,\dots,0}=0.$$

Hence, there exists a system $(A_0, A_1, \dots, A_{r-1})$ of linear subspaces A_i of $\mathbf{P}^n$ such that

(2) $$f(Z)\cap\Omega_{n-r,n-r+1,\dots,n-r+r'-1,n-r+r'+1,n-r+r'+2,\dots,n}(A_0, A_1,\dots, A_{r-1}) =\phi.$$

We fix a $(n-r+r')$-dimensional linear subspace A of $\mathbf{P}^n$, which contains $A_{r'-1}$. For any point x of $f(Z)$, we have

$\dim(L_x\cap A)\geqq r'-1$ and $\dim(L_x\cap A_{r'-1})<r'-1$ (by virtue of (2)). Hence we have

$$\dim(L_x\cap A)=r'-1.$$

Therefore, we can define a morphism $g: f(Z)\to \mathrm{Gr}(n-r+r', r'-1)$, by $L_{g(x)}=L_x\cap A\subset A\approx\mathbf{P}^{n-r+r'}$ for any point x of $f(Z)$. It is easy to see that

$$(g\circ f)(Z)\cdot\omega_{1,1,\dots,1}=0 \qquad (\text{in } \mathrm{Gr}(n-r+r', r'-1)).$$

Hence, by virtue of the proof of Lemma 2.4, there exists a curve C in Z, such that for any point y of $(g\circ f)(C)$, L_y passes through a common point. This shows that for any point x of $f(C)$, L_x passes through a common point, and this shows that $E|_C$ has a trivial line bundle as a direct summand. But this contradicts the fact that E is an ample vector bundle. Thus we proved that $c_{r'}(E)$ is numerically positive.

If $|I|=r'\leqq\min\{r, m\}$, it is easy to show that

$$\omega_{1,0,\dots,0}^{i_1}\cdot\omega_{1,1,0,\dots,0}^{i_2}\cdot\dots\cdot\omega_{1,1,\dots,1}^{i_r}$$
$$=\omega_{1,1,\dots,1,0,\dots,0}+\text{sum of other Schubert cycles of non-negative coefficient.}$$

This shows that $c^I(E)$ is numerically positive in this case. q.e.d.

§3. Morphisms from projective spaces to Gr(n, d)

In this section we are gonig to show that all morphisms from $\mathbf{P}^m$ to $\mathrm{Gr}(n, d)$ is constant if $m\geqq n+1$ or if $m=n\geqq 6$ and $d=1$ or 2.

Let m be an integer with $m\geqq n-d+1$ and assume that $n>2d>0$.

Let f be a morphism from $\mathbf{P}^m$ to $\mathrm{Gr}(n, d)$ and let $E=f^*\check{E}(n, d)$. Let c_i be the integer such that

$c_i(E)=c_i h^i$ where h is a hyperplane $(1 \leqq i \leqq d+1)$. Since E is generated by its global sectios, c_i is a non-negative integer.

Set $\quad c=(c_1, c_2, \dots, c_{d+1})$,

$$F(t)=1-c_1 t+c_2 t^2-\cdots+(-1)^{d+1} c_{d+1} t^{d+1} \quad \text{and}$$

$$G(t)=1+\Phi_1(c)t+\Phi_2(c)t^2+\cdots+\Phi_{n-d}(c)t^{n-d}.$$

$F(t)$ and $G(t)$ are elements of $\mathbf{Z}[t]$. Then, we have

Lemma 3.1. *Under the above notation, we have*

$$\Phi_{n-d+1}(c)=\Phi_{n-d+2}(c)=\cdots=\Phi_m(c)=0.$$

Proof. Since $f^*Q(n, d)$ is a vector bundle of rank $n-d$ and $c_i(f^*Q(n, d))=\Phi_i(c)h^i$, the assersion is obvious.

Corollary 3.2. *Assume that $m \geqq n+1$ and f be a morphism from $\mathbf{P}^m$ to $\mathrm{Gr}(n, d)$. Then $f(\mathbf{P}^m)$=one point.*

Proof. We may assume that $m=n+1$. We use same notation as above. By virtue of Lemma 3.1. we have

$$F(t)\cdot G(t)=1.$$

Therefore, we have

$$c_1=c_2=\cdots=c_{d+1}=0.$$

In particular we have $c_1(E)=0$. This shows that

$$f(\mathbf{P}^{n+1})\cdot\omega_{1,0,\dots,0}=0.$$

Since $\omega_{1,0,\dots,0}$ is an ample divisor, this shows that $f(\mathbf{P}^{n+1})$ is one point.

q.e.d.

Lemma 3.3. *If $m=n$ and $f(\mathbf{P}^m)$ is not one point, then*

(i) $c_1, c_2, \dots, c_{d+1}, \Phi_1(c), \Phi_2(c), \dots, \Phi_{n-d}(c)$ are positive integers.

(ii) *Set* $r=\mathrm{M.C.D.}(i, d+1)$ *and set* μ, γ *be such that* $i=r\mu$ *and* $d+1=r\gamma$. Then $c_i^\gamma c_{d+1}^{-\mu}$ *is a positive integer less than* $\binom{d+1}{i}^\nu$, *for all* i *with* $1 \leqq i \leqq d+1$.

(iii) *When nd is even, there exists an integer a such that*

$$c_{d+1}=a^{d+1}.$$

When nd is odd, there exists an integer a xuch that

$$c_{d+1}=a^s \text{ where } 2s=d+1.$$

Proof. By virtue of Lemma 3.1, we have

$$F(t)\cdot G(t)=1+(-1)_{d+1}c_{d+1}\cdot\Phi_{n-d}(c)t^{n+1}. \tag{1}$$

By the same way as in the proof of Corollary 3.2, we have

$$c_1>0 \quad \text{and} \quad c_{d+1}\Phi_{n-d}(c)\neq 0. \tag{2}$$

Hence, by virtue of formula (2) and Proposition 2.1, we have (i).

(ii): Let β be the positive $(n+1)$-st root of $c_{d+1}\Phi_{n-d}(c)$.

Set

$$F(t)=(1-\alpha_1\beta t)(1-\alpha_2\beta t)\dots(1-\alpha_{d+1}\beta t),$$

By virtue of formula (1), we have

$$|\alpha_i|=1, \alpha_i\neq\alpha_j \quad \text{if} \quad i\neq j, \quad \alpha_i^{-1}=\bar{\alpha}_i\in\{\alpha_1, \alpha_2, \dots, \alpha_{d+1}\}, \tag{3}$$

$$\alpha_1\cdot\alpha_2\dots\alpha_{d+1}=1 \quad \text{and} \quad \beta^{d+1}=c_{d+1}.$$

$$c_i^\gamma c_{d+1}^{-\mu}=\left(\sum_{1\leqq j_1<j_2<\dots<j_i\leqq d+1}\alpha_{j_1}\alpha_{j_2}\dots\alpha_{j_i}\right)^\nu$$
$$<\binom{d+1}{i}^\nu.$$

Hence, we have

Since $c_i^\gamma c_{d+1}^{-\mu}$ is a rational number and is integral over **Z**, it is a rational integer.

(iii): By virtue of the formula (3), we have

$$\begin{aligned}F\left(\frac{1}{\beta t}\right)t^{d+1} &= (t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_{d+1})\\ &= (\alpha_1^{-1}t-1)(\alpha_2^{-1}t-1)\dots(\alpha_{d+1}^{-1}t-1)\alpha_1\alpha_2\dots\alpha_{d+1}\\ &= (\alpha_1 t-1)(\alpha_2 t-1)\dots(\alpha_{d+1}t-1)\\ &= (-1)^{d+1}F\left(\frac{t}{\beta}\right).\end{aligned}$$

This shows that

$$(4) \qquad c_i\beta^{-i} = c_{d+1-i}\beta^{-d-1+i} \quad \text{for all } i \text{ with } 1 \leqq i \leqq d.$$

Case 1. When d is even. Let $d=2m$, then by virtue of the formula (4), we have

$$\beta = c_{m+1}c_m^{-1}.$$

Since β is a rational number and is integral over $\mathbf{Z}$, β is a rational integer. Let $a=\beta$, then we have $c_{d+1}=a^{d+1}$.

Case. 2. When d is odd and n is even. Siπe $n-d$ is odd, we can apply similar technic to $G(t)$ as in the Case 1 to $F(t)$, and we see that β is an integer. Let $a=\beta$, then we have

$$c_{d+1} = a^{d+1}.$$

Case 3. When d and n are odd. Let $d+1=2s$, then by virtue of the formula (4), we have

$$\beta^2 = c_{s+1}c_{s-1}.$$

Hence, β^2 is an integer. Let $a=\beta^2$, then we have $c_{d+1}=a^s$.

q.e.d.

Proposition 3.4. *Let* $n \geqq 6$ *and let* f *be a morphism from* $\mathbf{P}^n$ *to* $\mathrm{Gr}(n, 1)$, *then* $f(\mathbf{P}^n)$ *consists of one point.*

Proof. Suppose that $f(\mathbf{P}^n)$ has more than one point. By virtue of Lemma 3.3, we see that $c_1^2c_2^{-1}$ is a positive integer less than 4. When $c_1^2c_2^{-1}=1$, we have $\Phi_2(c)=0$. When $c_1^2c_2^{-1}=2$, we have $\Phi_3(c)=0$. When $c_1^2c_2^{-1}=3$, we have $\Phi_5(c)=0$. Since $n\geqq 6$, this contradicts (i) of Lemma 3.3. q.e.d.

Proposition 3.5. *Let $n\geqq 6$ and let f be a morphism from $\mathbf{P}^n$ to* Gr$(n, 2)$, *then $f(\mathbf{P}^n)$ consists of one point.*

Proof. Suppose $f(\mathbf{P}^n)$ has more than one point. By virtue of Lemma 3.3, we can write

$F(t)=(1-at)(1-bat+a^2t^2)$ where b and a are integers. By the same way as in the proof of Lemma 3.3, we see that b^2 is less than 4. When $b=-1$, we have $F(t)=1-a^3t^3$. This contradict (i) of Lemma 3.3. When $b=0$, we have $F(t)=1-at+a^2t^2-a^3t^3$. Hence, we have $\Phi_2(c)=0$. This contradict (i) of Lemma 3.3. When $b=1$, we have $F(t)=1-2at+2a^2t^2-a^3t^3$. Hence, we have $\Phi_4(c)=0$. This contradict (i) of Lemma 3.3. q.e.d.

§4. Numerically property of $(n-1)$-dimensional projective space in Gr$(n, 1)$

Let X be a subvraiety of Gr$(n, 1)$ which is biregular to $\mathbf{P}^{n-1}$, H a hyperplane of $X\approx\mathbf{P}^{n-1}$ and $E=\check{E}(n, 1)|_X$. Set

$$c_1(E)=X\cdot\omega_{1,0}=hH$$

$$c_2(E)=X\cdot\omega_{1,1}=bH^2 \qquad \text{(as cycles in } X\approx\mathbf{P}^{n-1}).$$

Then, we call that the triple (h, b, n) and the vector bundle E are associated with X.

In the sequel we shall say that a triple (h, b, n) is *admissible* if and only if the triple $(h. b, n)$ is associated with X, for some suitable subvariety X of Gr$(n, 1)$, which is biregular to $\mathbf{P}^{n-1}$.

The aim of this section is to prove the following theorem.

Theorem 4.1. *Assume that a triple (h, b, n) is admissible and $b\neq 0$, then*

(i) *when* $n=3$, $(h, b)=(1, 1)$ *or* (2.1) *or* $(2, 3)$

(ii) *when* $n=4$, $(h, b)=(2, 1)$ *or* $(2, 2)$

(iii) *when* $n\geqq 5$, $(h, b)=(2, 1)$.

In order to prove Theorem 4.1, we need some prerilinaries.

Lemma 4.2. *In* $\mathrm{Gr}(n, 1)$, *we have*

(i) $\omega_{i,j}\cdot\omega_{1,1}=\omega_{i+1,j+1}$

(ii) *let* $n-1\geqq i\geqq j\geqq 0$, $n-1\geqq k\geqq m\geqq 0$ and $i+j+k+m=2n-2$,

then,

$$\omega_{i,j}\omega_{k,m}=\begin{cases} 1 & \textit{if } i+m=j+k=n-1 \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Since $\omega_{1,1}=\omega_{1,0}{}^2-\omega_{2,0}$ and $\omega_{n-1,n-1}=$one point, the assertion is proved by easy calculation.

Lemma 4.3. *Let X be a subvariety of* $\mathrm{Gr}(n, 1)$ *which is biregular to* $\mathbf{P}^{n-1}$ *and let N be the normal bundle of X in* $\mathrm{Gr}(n, 1)$. *Assume that (h, b, n) is the triple associated with X. Then,*

$$c_{n-1}(N)=\sum_{i=0}^{[(n-1)/2]}(\Phi_{n-1-2i}(h, b)b^i)^2.$$

where $[(n-1)/2]$ *is the integer part of* $(n-1)/2$.

Proof. By virtue of Lemma 4.2, we have X is rationally equivalent to $\sum_{i=0}^{[(n-1)/2]}(X\cdot\omega_{n-1-i,i})\cdot\omega_{n-1-i,i}$. By virtue of [1], [9], we have

$$c_{n-1}(N)=X\cdot X=\sum_{i=0}^{[(n-1)/2]}(X\cdot\omega_{n-1-i,i})^2.$$

On the other hand we obtain

$$X\cdot\omega_{n-1-i,i}=X\cdot\omega_{n-1-2i,0}\omega_{i,i} \qquad \text{(by virtue of Lemma 4.2)}$$

$$=X\cdot\Phi_{n-1-2i}(\omega_{1,0},\ \omega_{1,1})\omega_{1,1}{}^i$$

$$=\Phi_{n-1-2i}(h,\ b)b^i. \qquad \text{q.e.d.}$$

Set $c_{n-1}(h,\ b)=\sum_{i=0}^{[(n-1)/2]}(\Phi_{n-1-2i}(h,\ b)b^i)^2$ and $f_n(h,\ b,\ t)=(1+ht+bt^2)^{n+1}/(1+(4b-h^2)t^2)(1+t)^n\in\mathbf{Z}[[t]]$. Then, we have the following lemma. q.e.d.

Lemma 4.4. *Assume that a triple* $(h,\ b,\ n)$ *is admissible and* $b\neq 0$, *then*

(i) *h and b are positive integers with* $h^2\geqq b$.

(ii) $\chi_i(f_n(h,\ b,\ n))>0$ *if* $0\leqq i\leqq n-1$.

(iii) $\chi_{n-1}(f_n(h,\ b,\ t))=c_{n-1}(h,\ b)$.

(iv) $\Phi_i(h,\ b)\geqq 0$ *if* $0\leqq i\leqq n-1$.

Proof. (i): Since $\omega_{1,0}{}^2-\omega_{1,1}=\omega_{2,0}$, we have

$$(h^2-b)H^2=(\omega_{1,0}{}^2-\omega_{1,1})\cdot X=\omega_{2,0}\cdot X\geqq 0 \quad\text{and}\quad h^2\geqq b.$$

Since $b\neq 0$, h and b are positive integers by virtue of Proposition 2.1.

(ii) and (iii): Let X be a subvariety of $\mathrm{Gr}(n,\ 1)$ with which the triple $(h,\ b,\ n)$ is associated. And let T_X (resp. $T_{\mathrm{Gr}(n,1)}$) be the tangent bundle of X (resp. $\mathrm{Gr}(n,\ 1)$). Then, there exist the following exact sequences of vector bundles,

$$0\longrightarrow T_X\longrightarrow T_{\mathrm{Gr}(n,1)}|_X\longrightarrow N\longrightarrow 0$$

$$0\longrightarrow \check{E}(n,\ 1)\otimes E(n,\ 1)\longrightarrow \overset{n+1}{\oplus}\check{E}(n,\ 1)\longrightarrow T_{\mathrm{Gr}(n,1)}\longrightarrow 0.$$

Hence, we have

$$c(N)=c(E)^{n+1}/c(E\otimes\check{E})c(T_X) \qquad \text{where} \quad E=\check{E}(n,\ 1)|_X.$$

Since $c(E)=X+hH+bH^2$, $c(E\otimes\check{E})=X+(4b-h^2)H^2$ and $c(T_X)=(X+H)^n$, we have

$$c(N)=(X+hH+bH^2)^{n+1}/(X+(4b-h^2)H^2)(X+H)^n.$$

By virtue of Lemma 4.3, we have (iii). It is easy to see that $\chi_1(f_n(h, b, t))>0$ and $\chi_{n-1}(f_n(h, b, t))>0$. Since $\check{E}(n, 1)$ is generated by its globalsections, so is N. By the descending induction on i and by virtue of Proposition 2.1, we can prove (ii). (iv): Since E is generated by its global sections, we have (iv) by virtue of Proposition 2.1.

q.e.d.

Lemma 4.5. *Let α and β be the cmplex number such that $1-ht+bt^2=(1-\alpha\sqrt{b}\,t)(1-\alpha\sqrt{b}\,t)$. Then,*

(i) $\Phi_m(h, b)=(\alpha^m+\alpha^{m-1}\beta+\cdots+\alpha\beta^{m-1}+\beta^m)\sqrt{b}^{\,m}$

(ii) $\Phi_m(h, b)=((\alpha^{m+1}-\beta^{m+1})/(\alpha-\beta))\sqrt{b}^{\,m}$

$$=(\sin(m+1)\theta(h, b)/\sin\theta(h, b))\sqrt{b}^{\,m} \qquad \text{if} \quad b\leqq h^2<4b$$

where $0<\theta(h, b)=\cos^{-1}(h/2b)\leqq\pi/3$.

Proof.
$$\begin{aligned}\Phi_m(h, b)&=\chi_m(1/(1-ht+bt^2))\\&=\chi_m(1/(1-\alpha\sqrt{b}\,t)(1-\beta\sqrt{b}\,t))\\&=(\alpha^m+\alpha^{m-1}\beta+\cdots+\alpha\beta^{m-1}+\beta^m)\sqrt{b}^{\,m}.\end{aligned}$$

(ii) is confirmed by easy calculation. q.e.d.

Lemma 4.6. *Assume that a triple (h, b, n) is admissible and $h^2<4b$. And set $n(h, b)=[\pi/\theta(h, b)]$, then we have*

$$n(h, b)\geqq n.$$

Proof. By virtue of (iv) of Lemma 4.4 and (ii) of Lemma 4.5, we have the result. q.e.d.

Lemma 4.7. *Assume that a triple (h, b, n) is admisseble, and $b\neq 0$. If $n=3$, then $(h, b)=(1, 1)$ or $(2, 1)$ or $(2, 3)$.*

Proof. Since $\chi_2(f_3(h, b, t))=7h^2-12h+5$ and since

$$c_2(h, b)=(h^2-b)^2+b^2,$$

we have

$14h^2-24h+12\geqq h^4$, by virtue of (iii) of Lemma 4.4. Then, we have $h=2$ or 1, which implies our assersion. q.e.d.

Lemma 4.8. *Assume that a triple (h, b, n) is admissible, and $b\neq 0$ If $n=4$, then $(h, b)=(2, 1)$ or $(2, 2)$.*

Proof. Since $\chi_3(f_4(h, b, t))=15h^3-44h^2+50h-20-4b\leqq 15h^3$ and since

$$c_3(h, b)=(h^3-2hb)^2+(hb)^2\geqq h^6/5$$

we have $75\geqq h^3$ and $4\geqq h$. Therefore, it is easy to see that $(h, b)=(2, 1)$ or $(2, 2)$. q.e.d.

Lemma 4.9. *Assume that a triple (h, b, n) is admissible, and $b\neq 0$. If $n=5$, then $(h, b)=(2, 1)$.*

Proof. Since

$$\chi_4(f_5(h, b, t))=31h^4-2h^2b+7b^2-5(26h^3+6hb)+15(16h^2+2b)$$
$$-210h+70\leqq 31h^4-2h^2b+7b^2\leqq 36h^4 \quad \text{and since}$$

$$c_4(h, b)=(h^4-3h^2b+b^2)^2+(h^2-b)^2b^2+b^4,$$

we have $36h^4\geqq b^4$ and $6h^2\geqq b^2$.

When $h^2\geqq 4b$, we have $c_4(h, b)\geqq h^8/16$. Thus we have $36\cdot 16\geqq h^4$ and $4\geqq h$. Hence, in this case $9\geqq b$.

Assume now that $h^2\leqq 4b$. Since $6h^2\geqq b^2$, we have $9\geqq h$ and $24\geqq b$. Therefore, it is easy to see that $(h, b)=(2, 1)$ q.e.d.

Lemma 4.10. *Assume that a triple (h, b, n) is admissible. If $n\geqq 6$, then 12 divides $hb(h^2-b+3)$.*

Proof. Let Y ($\approx \mathbf{P}^5$) be a linear subspace of dimension 5 of $X \approx \mathbf{P}^{n-1}$, then by Riemann-Roch Theorem, we have

$$\chi(E|_Y)=\chi_5\Big(\Big(1+3t+\frac{17}{4}t^2+\frac{15}{4}t^3+\frac{137}{60}t^4+t^5\Big)\Big(1+ht+\frac{1}{2}(h^2-2b)t^2$$

$$+\frac{1}{6}(h^3-3hb)t^3+\frac{1}{24}(h^4-4h^2b+2b^2)t^4+\frac{1}{120}(h^5-5h^3b+5hb^2)t^5\Big)\Big)$$

$$=1+\chi(\mathcal{O}(h))-4b-2hb-\left\{\frac{1}{2}h^2b-\frac{1}{4}b(b+1)+\frac{1}{24}hb(h^2-b+3)\right\}.$$

Since $\chi(E|_Y)$ and $\chi(\mathcal{O}(h))$ are integers, 12 has to divides $hb(h^2-b+3)$.

q.e.d.

Lemma 4.11. *If $n \geqq 6$ and $h^2<4b$, then there exists no admissible triple (h, b, n).*

Proof. Since $n \geqq 6$ and $h^2<4b$, we have $n(h, b) \geqq 6$. Hence, we have

(1) $$3b \leqq h^2 < 4b.$$

We also have

(2) $c_{n-1}(h, b) \geqq 6b^{n-1}$, by virtue of Lemma 4.5 (ii).

On the other hand we have

(3) $$\chi_{n-1}((1+ht+bt^2)^{n+1})=\chi_{n-1}(f_n(h, b, t)(1+t)^n(1+(4b-h^2)t^2)$$

$$>\chi_{n-1}(f_n(h, b, t))$$

$$=c_{n-1}(h, b) \text{ by virtue of Lemma 4.4.}$$

and

(4) $$\chi_{n-1}((1+ht+bt^2)^{n+1})<\chi_{n-1}((1+\sqrt{b}t)^{2n+2})$$

$$=\binom{2n+2}{n-1}\sqrt{b}^{\,n-1}$$

Since $\binom{2n+4}{n} \leqq 4\binom{2n+2}{n-1}$ (when $n \geqq 6$), we have

(5) $$\binom{2n+2}{n-1} \leqq 2 \cdot 4^{n-1}.$$

Therefore, by (2), (3), (4) and (5), we have

(6) $$15 \geqq b \quad \text{and} \quad 7 \geqq h.$$

The following table is that of $n(h, b)$ with the pair (h, b) which satisfies the condition (1) and (6).

(7)

(h, b)	$n(h, b)$	(h, b)	$n(h, b)$
(3, 3)	6	(6, 11)	7
(4, 5)	6	(6, 12)	6
(5, 7)	9	(7, 13)	12
(5, 8)	6	(7, 14)	8
(6, 10)	9	(7, 15)	7

In the pairs (h, b) which appear in the table (7), only (6, 10), (6, 11) and (6, 12) satisfy the condition of Lemma 4.11.

The following table shows that there exists no admissible triple (h, b, n) with $n \geqq 6$ and $h^2 < 4b$.

When $h=6$.

b	n	$\chi_{n-1}((1+ht+bt^2)^{n+1})$	$c_{n-1}(h, b)$
10	6	52, 8696	$\geqq 6 \cdot 10^5$
10	7	650, 3168	$\geqq 10 \cdot 10^6$
10	8	7950, 8736	$\geqq 10 \cdot 10^7$
10	9	9, 6855, 3120	$\geqq 10 \cdot 10^8$
11	6	57, 2166	$\geqq 6 \cdot 11^5$
11	7	720, 2104	$\geqq 6 \cdot 11^6$
12	6	61, 6896	$\geqq 6 \cdot 12^5$

(cf. Lemma 4.5. (ii)) q.e.d.

Lemma 4.12. *If* $n \geqq 6$ *and* $h^2 \geqq 4b$, *then* $(2, 1, n)$ *is the only admissible triple with* $b \neq 0$.

Proof. Let α be a positive number such that

$$1+ht+bt^2=(1+\alpha\sqrt{b}\,t)\Big(1+\frac{1}{\alpha}\sqrt{b}\,t\Big) \quad \textit{with} \quad \alpha \geqq 1 .$$

Then by virtue of Lemma 4.5, we have

$$(8) \qquad c_{n-1}(h, b)>\Phi_{n-1}(h, b)^2>(\alpha^{n-1}+\alpha^{n-3})^2 b^{n-1}=\Big(1+\frac{1}{\alpha^2}\Big)^2(\alpha^2 b)^{n-1} .$$

On the other hand we have

$$\begin{aligned}(9) \qquad &\chi_{n-1}((1+ht+bt^2)^{n+1}/(1+(4b-h^2)t^2))\\ &=\chi_{n-1}(f_n(h, b, t)(1+t)^n)\\ &>\chi_{n-1}(f_n(h, b, t))\\ &=c_{n-1}(h, b) \text{ by virtue of Lemma 4.4}\end{aligned}$$

and

$$\begin{aligned}(10) \qquad &\chi_{n-1}((1+ht+bt^2)^{n+1}/(1+(4b-h^2)t^2))\\ &=\chi_{n-1}\Big((1+\alpha\sqrt{b}\,t)^{n+1}\Big(1+\frac{1}{\alpha}\sqrt{b}\,t\Big)^{n+1}\Big(1-\Big(\alpha-\frac{1}{\alpha}\Big)^2 bt^2\Big)^{-1}\Big)\\ &=\sum_{i+j+2k=n-1}\binom{n+1}{i}\binom{n+1}{j}\alpha^{i-j}\Big(\alpha-\frac{1}{\alpha}\Big)^{2k} b^{n-1}\\ &\leqq\Big\{\sum_{j=0}^{n-1}\binom{n+1}{j}\alpha^{-2j}\Big(\sum_{i+2k=n-1-j}\binom{n+1}{i}\Big)\Big\}(\alpha\sqrt{b})^{n-1}\\ &\leqq\Big(1+\frac{1}{\alpha^2}\Big)^{n+1}2^n(\alpha\sqrt{b})^{n-1} .\end{aligned}$$

By (8), (9) and (10), we have

$$2^n\Big(1+\frac{1}{\alpha^2}\Big)^{n-1}>(\alpha\sqrt{b})^{n-1} .$$

Since $\sqrt[n-1]{2} \leqq \sqrt[5]{2}<1.15$, we have

$$2.3\left(1+\frac{1}{\alpha^2}\right)>\alpha\sqrt{b}\,. \tag{11}$$

Since $\left(1-\frac{1}{\alpha^2}\right)\alpha\sqrt{b}=\sqrt{h^2-4b}$, we have

$$2.3>2.3\left(1-\frac{1}{\alpha^4}\right)>\sqrt{h^2-4b}\,.$$

Therefore, we have $h^2-4b=0$ or 1 or 4 or 5. In the case when $h^2-4b=5$, (h, b) does not satisfy the condition of Lemma 4.10. Only $(h, b)=(6, 9), (4, 4), (2, 1), (7, 12), (5, 6), (3, 2)$ and $(4, 3)$ satisfy (11) and $h^2-4b=0$ or 1 or 4.

Case 1. When $(h, b)=(3, 2)$. We have

$$\chi_{n-1}(f_n(3, 2, t))=\chi_{n-1}((1+2t)^{n+1}(1-t)^{-1})<3^{n+1} \quad \text{and}$$

$$c_{n-1}(3, 2)>\Phi_{n-1}(3, 2)^2>(2^{n-1}+2^{n-2}+2^{n-3})^2>3\cdot 3^{2n-2}\,.$$

Hence, a triple $(3, 2, n)$ is not admissible if $n\geqq 6$.

Case 2. When $(h, b)=(4, 3)$. We have

$$\chi_{n-1}(f_n(4, 3, t))=\chi_{n-1}((1+3t)^{n+1}(1+t)(1-4t^2)^{-1})<\frac{1}{4}\cdot 6^{n+1} \quad \text{and}$$

$$c_{n-1}(4, 3)>\Phi_{n-1}(4, 3)^2>(3^{n-1}+3^{n-2}+3^{n-3})^2>2\cdot 9^{n-1}\,.$$

Hence, a triple $(4, 3, n)$ is not admissible if $n\geqq 6$.

Case 3. When $(h, b)=(7, 12)$. We have

$$\begin{aligned}\chi_{n-1}(f_n(7, 12, t))&<\chi_{n-1}((1+7t+12t^2)^{n+1}(1-t^2)^{-1})\\ &<\chi_{n-1}((1+4t)^{2n+2}(1-t^2)^{-1})\\ &<4^{n-1}\sum_{i=0}\binom{2n+2}{n-1-2i}\\ &<4^{n-1}\cdot\frac{1}{4}\cdot 2^{2n+2}=4^{2n-1}\,.\end{aligned}$$

On the other hand we have

$$c_{n-1}(7, 12)>\Phi_{n-1}(7, 12)^2=(4^n-3^n)^2=4^n\left(1-\left(\frac{3}{4}\right)^2\right)^2>4^{n-1}\,.$$

Hence, a triple (7, 12, n) is not admissible if $n \geqq 6$.

Case 4. When $(h, b)=(5, 6)$. We have

$$\begin{aligned}\chi_{n-1}(f_n(6, 5, t)) &= \chi_{n-1}((1+2t)^{n+1}(1+3t)^{n+1}(1-t^2)^{-1}(1+t)^{-n})\\
&= \chi_{n-1}\Big(\sum_{i+j\leqq n-1}(1+t)^{-1}\binom{n+1}{i}\binom{n+1}{j}\times\\
&\qquad (1+t)^{n+1-i-j}2^j t^{i+j}\Big)\\
&< \sum_{j\leqq n-1}\binom{n+1}{j}2^{n+1}+\sum_{j\leqq n-2}\binom{n+1}{1}\binom{n+1}{j}2^n\\
&\quad +\sum_{j\leqq n-3}\binom{n+1}{2}\binom{n+j}{1}2^{n-1}+\sum_{\substack{i+j\leqq n-1\\ i\geqq 3}}\binom{n+1}{2}\binom{n+1}{j}2^{n-2}\\
&< 4^{n+1}+(n+1)2^{2n+1}+n(n-1)2^{2n}+2^{3n}.\end{aligned}$$

On the other hand we have

$$\begin{aligned}c_{n-1}(5, 6) &> \Phi_{n-1}(5, 6)^2+\Phi_{n-3}(5, 6)^2 6^2\\
&= (3^n-2^n)^2+(3^{n-2}-2^{n-2})^2 6^2\\
&\geqq \frac{2}{3}\cdot 3^{2n}+\frac{2}{3}\cdot 3^{2n-4}6^2.\end{aligned}$$

Hence, it is easy to see that

$$\chi_{n-1}(f_n(5, 6, t))<c_{n-1}(5, 6).$$

Therefore, a triple (5, 6, n) is not admissible if $n \geqq 6$.

Case 5. When $(h, b)=(2, 1)$. We have

$$\chi_{n-1}(f_n(2, 1, t))=\chi_{n-1}((1+t)^{n+2})=\frac{1}{6}n(n+1)(n+2).$$

On the other hand we have

$$\begin{aligned}c_{n-1}(2, 1) &= \sum_{i=0}^{[(n-1)/2]}\Phi_{n-1-2i}(2, 1)^2\\
&= \sum_{i=0}^{[(n-1)/2]}(n-2i)^2=\frac{1}{6}n(n+1)(n+2).\end{aligned}$$

Hence, a triple $(2, 1, n)$ satisfies the condition (iii) of Lemma 4.4 for any n. It is easy to see that a triple $(2, 1, n)$ satisfies other conditions of Lemma 4.4.

Case 6. When $(h, b)=(4, 4)$ or $(6, 9)$. Let $h=2(a+1)$ and $b=(a+1)^2$ where $a=1$ or 2. Then, we have

$$\chi_{n-1}(f_n(h, b, t))=\chi_{n-1}((1+(a+1)t)^{2n+2}(1+t)^{-n})$$

$$=\chi_{n-1}\left(\sum_{i=0}^{n-1}\binom{2n+2}{i}(1+t)^{n+2-i}a^i t^i\right)$$

$$=\sum_{i=0}^{n-1}\binom{2n+2}{i}\binom{n+2-i}{n-1-i}a^i$$

On the other hand we have

$$c_{n-1}(h, b)=b^{n-1}c_{n-1}(2, 1)$$

$$=(a+1)^{2n-2}\binom{n+2}{3}$$

$$=(1+a)^{-1}(1+a)^{2n-1}\binom{n+2}{3}$$

$$=\sum_{i=0}^{n-1}(1+a)^{-1}(1+a^{2n-1-2i})a^i\binom{2n-1}{i}\binom{n+2}{3}$$

$$\geqq\sum_{i=0}^{n-1}\binom{2n-1}{i}\binom{2+2}{3}a^i.$$

Since $\binom{2n-1}{i}\binom{n+2}{3}-\binom{2n+2}{i}\binom{n+2-i}{n-1-i}\geqq 0$ ($=0$ if and only if $i=0$),

$$c_{n-1}(h, b)>\chi_{n-1}(f_n(h, b, t)).$$

Hence, triples $(4, 4, n)$ and $(6, 9, n)$ are not admissible if $n\geqq 6$.

q.e.d.

By virtue of Lemma 4.7, Lemma 4.8, Lemma 4.9, Lemma 4.11 and Lemma 4.12, Theorem 4.1 is proved.

§5. $(n-1)$-dimensional projective spaces in $\mathrm{Gr}(n, 1)$

Let $(x_0, x_1,\dots, x_n)$ be a system of homogeneous coordinates in $\mathbf{P}^n$ When

$$A=\begin{pmatrix} a_{0,0}, & a_{0,1},\dots, & a_{0,n} \\ a_{1,0}, & a_{1,1},\dots, & a_{1,n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{d,0}, & a_{d,1},\dots, & a_{d,n} \end{pmatrix}$$

is a $(d+1, n+1)$ matrix of rank $d+1$, we denote by the same symbol A the element of $\mathrm{Gr}(n, d)$ which represents the d-space in $\mathbf{P}^n$ spanned by $(d+1)$-points $(a_{0,0}, a_{0,1},\dots, a_{0,n})$, $(a_{1,0}, a_{1,1},\dots, a_{1,n}),\dots, (a_{d,0}, a_{d,1},\dots, a_{d,n})$.

For any i with $0\leqq i\leqq d$, we define a morphism $f^i_{n,d}$ from $\mathbf{P}^{n-d}$ to $\mathrm{Gr}(n, d)$ by the following way.

$$f^i_{n,d}((x_0, x_1,\dots, x_{n-d}))=$$

$$\left(\begin{array}{cccccccccc}
1, & 0, & \dots, & 0, & & & & & & \\
0, & 1, & \dots, & 0, & & & & & & \\
\cdot & \cdot\ \cdot & & \cdot & & & 0 & & & \\
\cdot & \cdot & \cdot & \cdot & & & & & & \\
\cdot & \cdot & & \cdot\ \cdot & & & & & & \\
0, & 0, & \dots, & 1, & & & & & & \\
 & & & & x_0, & x_1, & \dots, & x_{n-d}, & & \\
 & & & & & x_0, & x_1, & \dots, & x_{n-d}, & 0 \\
 & 0 & & & & \cdot & \cdot & & \cdot & \\
 & & & & & & \cdot & \cdot & & \cdot \\
 & & & & & & & \cdot & \cdot & \cdot \\
 & & & & & & x_0, & x_1, & \dots, & x_{n-d}
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} n-i \\
\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} i+1
\end{array}$$

Let $X^i_{n,d}=f^i_{n,d}(\mathbf{P}^{n-d})$. It is easy to see that $X^i_{n,d}$ is biregular to $\mathbf{P}^{n-d}$ and that $X^i_{n,d}$ and $X^j_{n,d}$ are projectively equivalent to each other if and only if $i=j$.

As mentioned in §1, there exists a canonical dual biregular morphism $\phi\colon \mathrm{Gr}(n, d)\to \mathrm{Gr}(n, n-d-1)$. We denote $\phi(X^i_{n,d})$ by $\check{X}^i_{n,d}$. The aim of this section is to prove the following Main Theorem.

Theorem 5.1. *Let X be a subvariety of* $\mathrm{Gr}(n, 1)$ *which is biregular to* $\mathbf{P}^{n-1}$. *Then,*

(i) *When $n=3$, X is projectively equivalent to some one of* $X^0_{3,1}$, $X^1_{3,1}$, $\check{X}^0_{3,1}$ and $\check{X}^1_{3,1}$.

(ii) *When $n\geqq 5$, X is projectively equivalent to either $X^0_{n,1}$ or $X^1_{n,1}$.*

In order to prove Theorem 5.1, we need some preliminaries.

Lemma 5.2. *Let X be a subvariety of* $\mathrm{Gr}(n, 1)$ *which is biregular to* $\mathbf{P}^{n-1}$ $(n\geqq 3)$. *Assume that a triple $(h, 0, n)$ is associated with X. Then, X is promecitvely equivalent to $X^0_{n,1}$. Consequently $h=1$.*

Proof. Set $E=\check{E}(n, 1)|_X$. Since $c_2(E)=X\cdot\omega_{1,1}=0$, we have, by virtue of Lemma 2.4

$$E\approx\mathcal{O}_X\oplus \text{(line bundle)}.$$

Hence, there exists a point P in $\mathbf{P}^n$ such that

$$X\subset\Omega_{0,n}(P)=\{x\in \mathrm{Gr}(n, 1)|L_x\in P\}\approx\mathbf{P}^{n-1}.$$

Therefore, $X=\Omega_{0,n}(P)$ and this is projectively equivalent to $X^0_{n,1}$. q.e.d.

Lemma 5.3. *Let X be a subvariety of* $\mathrm{Gr}(n, 1)$ *which is biregular to* $\mathbf{P}^{n-1}$ $(n\geqq 3)$ *and let* $E=\check{E}(n, 1)|_X$. *Assume that the triple* $(2, 1, n)$ *is associated with X. Then*

$$E=\mathcal{O}_X(1)\oplus\mathcal{O}_X(1).$$

In order to proof Lemma 5.3, we need following lemmas.

Lemma 5.4. *Let E be a vector bundle of rank 2 on $\mathbf{P}^2$. Assume that E is not simple*[5] *and uniform vector bundle. Then, E is decomposable.* (cf. [11], *Theorem* 4.10 *or* [7])

Lemma 5.5. *Let E be an indecomposable and almost decomposable*[5] *vector bundle of rank* 2 *on a variety Y with* $\dim Y\geqq 2$. *Then* there exists a line bundle L such that

(i) $h^0(E\otimes L)=1$

(ii) $h^0(\check{E}\otimes\check{L})=h^0(\det(\check{E}\otimes\check{L}))>0.$

(Schwarzenberger [7]).

Proof of Lemma 5.3. When $n=3$. Since E is generated by its global sections, for any line ℓ in $X\approx\mathbf{P}^2$, $E|_\ell=\mathcal{O}_\ell(1)\oplus\mathcal{O}_\ell(1)$ or $E|_\ell=\mathcal{O}_\ell(2)\oplus\mathcal{O}_\ell$. Suppose that for any line ℓ in X, $E|_\ell=\mathcal{O}_\ell(2)\oplus\mathcal{O}_\ell$, i.e. E is uniform. Since $c_1(E)^2-4c_2(E)=0$, E is not simple (cf [7]). Hence, E is decomposable by virtue of Lemma 5.4. Hence, $E=\mathcal{O}_X(2)\oplus\mathcal{O}_X$. This contradict the fact that $c_2(E)=1$. Therefore, there exists some line ℓ in X, such that $E|_\ell=\mathcal{O}_\ell(1)\oplus\mathcal{O}_\ell(1)$.

We now assume that $n\geqq 3$. By induction on n and by the fact we proved in the above, we may assume that there exists a hyperplane H of $X\approx\mathbf{P}^{n-1}$, such that

$E|_H=\mathcal{O}_H(1)\oplus\mathcal{O}_H(1)$. Since, for any integer m, the sequence

$$0\longrightarrow E(m-1)\longrightarrow E(m)\longrightarrow\mathcal{O}_H(m+1)\oplus\mathcal{O}_H(m+1)\longrightarrow 0$$

is exact, it is easy to see that $h^0(E(m))=h^1(E(m))=0$ if $m\leqq -2$. Hence, we have $h_0(E(-1))=2$. Since $(E(-1)^\vee)=E(-1)\oplus\det(E(-1)^\vee)=E(-1)$, E is an almost decomposable vector bundle. Since $h^0(E(-1))=2$ and

5) Avector bundle E is said to be almost decomposable or not simple if and only if $\dim H^0(X, E\otimes\check{E})>1$.

$h^0(E(-2))=0$, it is easy to see that E is a decomposable vector bundle by virtue of Lemma 5.4. Therefore,

$$E=\mathcal{O}_X(1)\oplus\mathcal{O}_X(1). \qquad \text{q.e.d.}$$

Lemma 5.6. *Let X be a subvariety of* $\mathrm{Gr}(n, 1)$ *which is biregular to* $\mathbf{P}^{n-1} (n\geqq 3)$. *Assume that the triple* $(2, 1, n)$ *is associated with X. Then X is projectively equivalent to $X^1_{n,1}$.*

Proof. Set $E=\check{E}(n, 1)|_X$. By virtue of Lemma 5.3, we have $E=\mathcal{O}_X(1)\oplus\mathcal{O}_X(1)$, whence X is given by the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1)\oplus\mathcal{O}_X(-1) \xrightarrow{\varphi} \overset{n+1}{\oplus}\mathcal{O}_X \longrightarrow (\text{quotient bundle}) \longrightarrow 0$$

This exact sequence is factered through φ'_i: $0 \longrightarrow \mathcal{O}_X(-1) \xrightarrow{\varphi'_i} \overset{n}{\oplus}\mathcal{O}_X$ $(i=1, 2)$ such that $\varphi=\varphi_1+\varphi_2$, where

$$\begin{tikzcd}
0 \arrow[r, dash] & \mathcal{O}_X(-1) \arrow[rr, "\varphi'_1"] \arrow[d, hook] \arrow[drr, "\varphi_1"] & & \overset{n}{\oplus}\mathcal{O}_X \arrow[d, hook] \\
0 \arrow[r, dash] & \mathcal{O}_X(-1)\oplus\mathcal{O}_X(-1) \arrow[rr, "\varphi"] & & \overset{n+1}{\oplus}\mathcal{O}_X \\
0 \arrow[r] & \mathcal{O}_X(-1) \arrow[u, hook] \arrow[rr, "\varphi'_2"] \arrow[urr, "\varphi_2"] & & \overset{n}{\oplus}\mathcal{O}_X \arrow[u, hook]
\end{tikzcd}$$

These φ_i gives a linear map

$$\psi_i\colon X\approx\mathbf{P}^{n-1} \longrightarrow \mathbf{P}^n \qquad (i=1, 2).$$

Since φ is injective, $\psi_1(P)\neq\psi_2(P)$ for any point P of X, and

$X=\{x\in \mathrm{Gr}(n, 1)|P\in X, L_x$ is a line passing through

$$\psi_1(P) \text{ and } \psi_2(P)\}.$$

Therefore, in order to cmoplete the proof, we have only to prove the following lemma.

Lemma 5.7. *Let A and B be hyperplanes of $\mathbf{P}^n$ and $\psi\colon A\to B$ be a linear map such that $P\neq\psi(P)$ for any point P of A. Then, we can choose suitable coordinate system of $\mathbf{P}^n$ such that*

(i) $A=\{$*points with* $x_n=0\}$ *and* $B=\{$*points with* $x_0=0\}$

(ii) $\psi((x_0, x_1,\dots, x_{n-1}, 0))=(0, x_0, x_1,\dots, x_{n-1})$.

In order to prove Lemma 5.7, we need the following definition of Δ-system and Lemma 5.8.

Definition. (I): *A system of six points of* $\mathbf{P}^n$ *expressed in the form*

$$\begin{array}{ccccc} & & P_0^2 & & \\ & P_0^1 & & P_1^1 & \\ P_0^0 & & P_1^0 & & P_2^0 \end{array}$$

is called a Δ*-system of size* 2 *if and only if*

(i) P_0^0, P_1^0 *and* P_2^0 *span a linear space of dimension* 2.

(ii) $P_0^1 \in \overline{P_0^0 P_1^0}$ *where* $\overline{P_0^0 P_1^0}$ *is the line passing through* P_0^0 *and* P_1^0 *and* $P_0^1 \neq P_0^0$ *and* $P_0^1 \neq P_1^0$*: similarly* $P_1^1 \in \overline{P_1^0 P_2^0}$ *and* $P_1^1 \neq P_1^0$ *and* $P_1^1 \neq P_2^0$.

(iii) $P_0^2 = \overline{P_0^1 P_2^0} \cap \overline{P_1^1 P_0^0}$.

(II): *When* $m \geqq 3$, *system of* $\binom{m+2}{2}$ *points of* $\mathbf{P}^n$ *expressed in the form*

$$\begin{array}{cccccc} & & P_0^m & & & \\ & & P_0^{m-1} & P_1^{m-1} & & \\ & & \cdot\ \ \cdot & & \cdot & \\ & \cdot\ \ \cdot & & & \cdot & \\ & P_0^1 & P_1^1 & \cdot & & P_{m-1}^1 \\ P_0^0 & P_1^0 & \cdot & & & P_m^0 \end{array}$$

or in the form $\{P_j^i | 0 \leqq i, j, i+j \leqq m\}$, *is called a* Δ*-system of size* m *if and only if*

(i) $P_0^0, P_1^0,\dots, P_m^0$ *span a linear space of dimension* m.

(ii)
$$\begin{array}{cccccc} & & P_0^{m-1} & & & \\ & & P_0^{m-2} & P_1^{m-2} & & \\ & & \cdot\ \ \cdot & & \cdot & \\ & \cdot\ \ \cdot & & & \cdot & \\ & P_0^1 & P_1^1 & \cdot & & P_{m-2}^1 \\ P_0^0 & P_1^0 & \cdot & & & P_{m-1}^0 \end{array} \quad \text{and} \quad \begin{array}{cccccc} & & P_1^{m-1} & & & \\ & & P_1^{m-2} & P_2^{m-2} & & \\ & & \cdot\ \ \cdot & & \cdot & \\ & \cdot\ \ \cdot & & & \cdot & \\ & P_1^1 & P_2^1 & \cdot & & P_{m-1}^1 \\ P_1^0 & P_2^0 & & & & P_m^0 \end{array} \quad \text{are}$$

Δ*-systems of size* $m-1$.

(iii) $\begin{matrix} & & P_0^m & & \\ & P_0^{m-1} & & P_1^{m-1} & \\ P_0^0 & & P_1^{m-2} & & P_2^0 \end{matrix}$ *is a Δ-system of size* 2.

Lemma 5.8. (I): *Let* $P_0^0, P_1^0, \dots, P_m^0; P_0^1, P_1^1, \dots, P_{m-1}^1$ *be* $2m+1$ *points of* $\mathbf{P}^n$ *such that*

(i) $P_0^0, P_1^0, \dots, P_m^0$ *span a linear space of dimension* m.

(ii) $P_i^1 \in \overline{P_i^0 P_{i+1}^0}$, $P_i^1 \neq P_i^0$ *and* $P_i^1 \neq P_{i+1}^0$ *for any* i *with* $0 \leqq i \leqq m-1$.

Then, there exists a completely determined one Δ-system $\{P_j^i | 0 \leqq i, j, i+j \leqq m\}$ *of size* m *such that* $P_0^0, P_1^0, \dots, P_m^0; P_0^1, P_1^1, \dots, P_{m-1}^1$ *are its bottom row and its second bottom row, respectively.*

(II): *If* $\{P_j^i | 0 \leqq i, j, i+j \leqq n\}$ *is a Δ-system of size* n *in* $\mathbf{P}^n$. *Then*

(i) $P_0^0, P_1^0, \dots, P_n^0, P_0^n$ *are in general position, i.e. any subset of* $n+1$ *points of* $\{P_0^0, P_1^0, \dots, P_n^0, P_0^n\}$ *spans* $\mathbf{P}^n$.

(ii) *if we choose coordinate system of* $\mathbf{P}^n$ *such that* $P_0^0 = (1, 0, \dots, 0)$, $P_1^0 = (0, 1, 0, \dots, 0), \dots, P_n^0 = (0, \dots, 0, 1)$ *and* $P_0^n = (1, 1, \dots, 1)$, *then* $P_0^{n-1} = (1, 1, \dots, 1, 0)$ *and* $P_1^{n-1} = (0, 1, \dots, 1)$

(III): *Let* A *and* B *be linear spaces of* $\mathbf{P}^n$ *and* $\psi: A \to B$ *be a linear map. Let* $\{P_j^i | 0 \leqq i, j, i+j \leqq m\}$ *be a Δ-system of size* m *in* A. *Then, the Δ-system of size* m *in* B *determined by* $2m \times 1$ *points* $Q_0^0 = \psi(P_0^0), Q_1^0 = \psi(P_1^0), \dots, Q_m^0 = \psi(P_m^0); Q_0^1 = \psi(P_1^1), Q_0^1 = \psi(P_1^1), \dots, Q_{m-1}^1 = \psi(P_{m-1}^1)$ *is* $\{Q_j^i = \psi(P_j^i) | 0 \leqq i, j, i+j \leqq m\}$.

Proof. It is easy. q.e.d.

Proof of Lemma 5.7. Let $A_0 = A$ and $B_0 = B$, and we define inductively $A_i = B_{i-1} \cap A_{i-1}$ and $B_i = \psi(A_i)$ $(1 \leqq i \leqq n-1)$. Since ψ has no fixed point, $A_i \neq B_i$. Hence,

$$\dim A_i = \dim B_i = n - 1 - i .$$

Set $P_n^0 = B_{n-1}$ (which is a point of $\mathbf{P}^n$). We define inductively $P_i^0 = \psi^{-1}(P_{i+1}^0)$. Then, it is easy to see that $P_i^0, P_{i+1}^0, \dots, P_n^0$ span B_{i-1}, that $P_0^0, P_1^0, \dots, P_{n-1}^0$ span A_0 and that $P_0^0, P_1^0, \dots, P_n^0$ span $\mathbf{P}^n$.

Fix a point P_0^0 on the line $\overline{P_0^0 P_1^0}$ such that $P_0^1 \neq P_0^0$ and $P_0^1 \neq P_1^0$, and we define inductively

$$P_i^1 = \psi(P_{i-1}^1) \quad \text{where} \quad 1 \leqq i \leqq n-1 .$$

Then, by virtue of Lemma 5.8, there exists a Δ-system $\{P_j^i | 0 \leqq i, j, i+j \leqq n\}$ of size n. Since $\psi(P_j^0) = P_{j+1}^0$ $(0 \leqq j \leqq n-1)$ and $\psi(P_j^1) = P_{j+1}^1$ $(0 \leqq j \leqq n-2)$, we have $\psi(P_0^{n-1}) = P_1^{n-1}$, by virtue of Lemma 5.8 (III). We choose a system of coordinate $\mathbf{P}^n$ such that $P_0^0 = (1, 0, \dots, 0)$, $P_1^0 = (0, 1, 0, \dots, 0), \dots, P_n^0 = (0, 0, \dots, 0, 1)$ and $P_0^n = (1, 1, \dots, 1)$. Then, we have $\psi((1, 1, \dots, 1, 0)) = (0, 1, 1, \dots, 1)$, by virtue of Lemma 5.8 (II). Therefore, we have

$$\psi((x_0, x_1, \dots, x_{n-1}, 0) = (0, x_0, x_1, \dots, x_{n-1}) . \qquad \text{q.e.d.}$$

Lemma 5.9. *Let X be a subvariety of* Gr(3, 1) *which is biregular to* $\mathbf{P}^n$.

(i) *Assume that the triple* (1, 1, 3) *is associated with X. Then, X is projectively equivalent to* $\check{X}_{3,1}^0$.

(ii) *Assume that the triple* (2, 3, 3) *is associated with X. Then, X is projectively equivalent to* $\check{X}_{3,1}^1$.

Proof. (i): Set $E = \check{E}(3, 1)|_X$. There exists an exact sequence of vector bundles

$$0 \longrightarrow \check{E} \longrightarrow \overset{4}{\oplus} \mathcal{O}_X \longrightarrow Q \longrightarrow 0$$

where $Q = Q(3, 1)|_X$. Then, X is the dual space to the space defined by the exact sequence

$$0 \longrightarrow \check{Q} \longrightarrow \overset{4}{\oplus} \mathcal{O}_X \longrightarrow E \longrightarrow 0 .$$

Since $c_2(Q) = \Phi_2(E) = 0$, we have the result by virtue of Lemma 5.2.

(ii): By the same way as in (i), we can prove (ii) q.e.d.

By virtue of Theorem 4.1, Lemma 5.2, Lemma 5.6 and Lemma 5.9, Theorem 5.1 is proved.

§6. On the family of lines lying on a non-singular quadric three fold

In this section we assume that the characteristic p of k is not equal to 2. Let S be a non-singular quadric hypersurface of $\mathbf{P}^4$, and let

$$X_q(S)=\{x\in \mathrm{Gr}(4, 1)|S\supset L_x\}\subset \mathrm{Gr}(4, 1).$$

In this section we shall show the following two theorems.

Theorem 6.1. (i) *$X_q(S)$ is biregular to $\mathbf{P}^3$.*

(ii) *$X_q(S)$ and $X_q(S')$ are projectively equivalent to each other, for any two non-singular quadric hypersurface S and S' of $\mathbf{P}^4$.*

Theorem 6.2. *Let X be a subvariety of* $\mathrm{Gr}(4, 1)$ *which is biregular to $\mathbf{P}^3$. Then, X is projectively equivalent to some one of $X^0_{4,1}$, $X^1_{4,1}$ and $X_q(S)$ where S is a non-singular quadric hypeersurface of $\mathbf{P}^4$.*

In order to prove Theorem 6.1, we need the following lemmas.

Lemma 6.3. *Let S be a non-singular quadric hypersurface of $\mathbf{P}^4$ and let L_1, L_2 and L_3 be three distinct lines which lie on S. Then,*

(i) *S contains no linear spaces of dimension* 2.

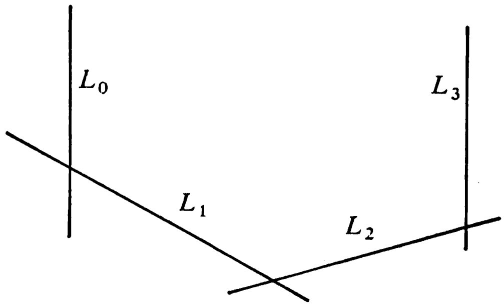
(ii) *No linear spaces of dimension* 2 *contain $L_1\cup L_2\cup L_3$.*

(iii) *Assume that L_1, L_2 and L_3 span $\mathbf{P}^4$ and $L_1\cap L_2\neq\phi$. Then, $L_1\cap L_3=\phi$ and $L_2\cap L_3=\phi$.*

(iv) *Assume that $L_1\cap L_2=\phi$, then for any point P on L_1, there exists only one point Q on L_2 such that $\overline{PQ}\subset S$ where $\overline{PQ}$ is the line which passes through P and Q.*

(v) *Assume that L_1 L_2 and L_3 span $\mathbf{P}^4$, then there exists only one line which lies on S and has a common point with every one of L_1, L_2 and L_3.*

(vi) *For any line L_0 which lies on S, there exist three distinct lines L_1, L_2 and L_3 which satisfy the following three conditions* (a), (b) and (c). (a): L_1, L_2 and L_3 lie on S. (b): L_0, L_1, L_2 and L_3 span $\mathbf{P}^4$. (c): $L_0 \cap L_1 \neq \phi$, $L_1 \cap L_2 \neq \phi$, $L_2 \cap L_3 \neq \phi$, $L_0 \cap L_2 = \phi$, $L_0 \cap L_3 = \phi$ and $L_1 \cap L_3 = \phi$.



Proof. (i): If S contains a linear space of dimension 2, S must have a singular point, which is not the case.

(ii): Since degree $S=2$, the proof is easy by virtue of (i).

(iii): It is obvious.

(iv): Let $H(L_1, L_2)$ be the hyperplane spanned by L_1 and L_2. It is easy to see that $S \cap H(L_1, L_2) \not\approx a$ cone and $S \cap H(L_1, L_2) \not\approx$ two planes. Hence, $S \cap H(L_1, L_2) \approx \mathbf{P}^1 \times \mathbf{P}^1$. And there exists an isomorphism f: $\mathbf{P}^1 \times \mathbf{P}^1 \to S \cap H(L_1, L_2)$ hwich satisfies the following conditions:

(α): $L_1 = f(\mathbf{P}^1 \times (1, 0))$ and $L_2 = f(\mathbf{P}^1 \times (0. 1))$,

(β): {lines lying on $S \cap H(L_1, L_2)$}

$$= \{f(\mathbf{P}^1 \times P) | P \in \mathbf{P}^1\} \cup (f(Q \times \mathbf{P}^1) | Q \in \mathbf{P}^1\}.$$

Hence, (iv) is obvious.

(v): Since L_1, L_2 and L_3 span $\mathbf{P}^4$, we may assume that $L_1 \cap L_2 = \phi$. Let f; $\mathbf{P}^1 \times \mathbf{P}^1 \to S \cap H(L_1, L_2)$ be as above. Since $L_3 \not\subset H(L_1, L_2)$, $L_3 \cap (H(L_1, L_2) \cap S) = L_3 \cap H(L_1, L_2)$ is one point, say $f((P_1, P_2))$, then $L = f(P_1 \times \mathbf{P}^1)$ is unique line which satisfies (v).

(vi): There exists a general hyperplane H such that $H \not\supset L_0$ and $S \cap H$ is a non-singular quadric surface. Hence, $S \cap H \approx \mathbf{P}^1 \times \mathbf{P}^1$. Since $L_0 \cap (S \cap H) = L_0 \cap H$ is one point, we can take two lines L_1

and L_3 such that L_1 and L_3 lie on $S \cap H$, $L_1 \cap L_3 = \phi$ and $L_0 \cap L_1 \neq \phi$. There exists a line L_2 which lies on $S \cap H$ and $L_1 \cap L_2 \neq \phi$, $L_2 \cap L_3 \neq \phi$ and $L_0 \cap L_2 = \phi$, by virtue of the proof of (iv). It is easy to see that the three distinct lines L_1, L_2 and L_3 satisfy the conditions (a), (b) and (c). q.e.d.

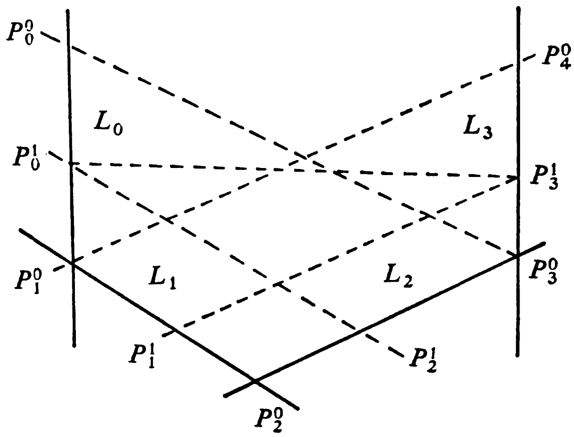
Lemma 6.4. *Let S and S' be two non-singular quadric hypersurfaces of $\mathbf{P}^4$ and let L_0, L_1, L_2, and L_3 (resp. L'_0, L'_1, L'_2 and L'_3) be four distinct lines which lie on S (resp. S'). Assume that L_0, L_1, L_2 and L_3 (resp. L'_0. L'_1, L'_2 and L'_3) span $\mathbf{P}^4$ and that $L_0 \cap L_1 \neq \phi$. $L_1 \cap L_2 \neq \phi$, $L_2 \cap L_3 \neq \phi$, $L_0 \cap L_2 = \phi$, $L_0 \cap L_3 = \phi$ and $L_1 \cap L_3 = \phi$ (resp. $L'_0 \cap L'_1 \neq \phi$, $L'_1 \cap L'_2 \neq \phi$, $L'_2 \cap L'_3 \neq \phi$, $L'_0 \cap L'_2 = \phi$, $L'_0 \cap L'_3 = \phi$ and $L'_1 \cap L'_3 = \phi$). Then,*

(i) *There exists a linear map $\sigma: \mathbf{P}^4 \to \mathbf{P}^4$ such that $\sigma(S) = S'$ and $\sigma(L_i) = L'_i$ for all $i = 0, 1, 2, 3$.*

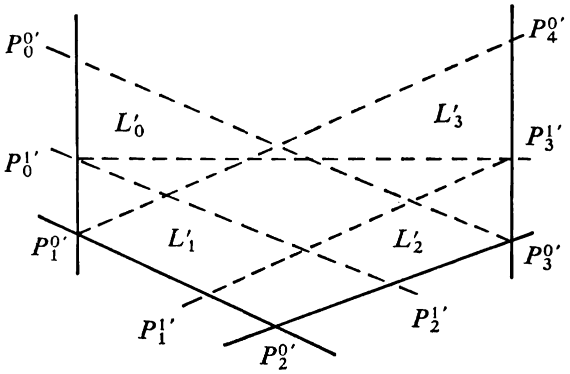
(ii) *In particular $X_q(S)$ and $X_q(S')$ are projectively eauivalent to each other.*

Proof. Set $P_1^0 = L_0 \cap L_1$, $P_2^0 = L_1 \cap L_2$ and $P_3^0 = L_2 \cap L_3$

Take P_0^0 the point on L_0 such that $\overline{P_0^0 P_3^0} \subset S$ and take P_4^0 the point on L_3 such that $\overline{P_1^0 P_4^0} \subset S$ (cf. Lemma 6.3. (iv)). Choose a point P_0^1 on L_0 such that $P_0^1 \neq P_0^0$ and $P_0^1 \neq P_1^0$. Then, there exist three points P_1^1, P_2^1 and P_3^1 such that $P_1^1 \in L_1$, $P_2^1 \in L_2$, $P_3^1 \in L_3$, $\overline{P_0^1 P_2^1} \subset S$, $\overline{P_0^1 P_3^1} \subset S$ and $\overline{P_1^1 P_3^1} \subset S$.



It is easy to see that 9 points $P_0^0, P_1^0, P_2^0, P_3^0, P_4^0; P_0^1, P_1^1, P_2^1, P_3^1$ satisfy the conditions of Lemma 5.8 (I). Hence, there exists a Δ-system $\{P_j^i | 0 \leqq i, j, i+j \leqq 4\}$ of size 4 such that $P_0^0, P_1^0, P_2^0, P_3^0, P_4^0$ form its bottom row and $P_0^1, P_1^1, P_2^1, P_3^1$ form its second bottom row. Similary we can define a Δ-system of size 4 $\{P_j^{i\prime} | 0 \leqq i, j, i+j \leqq 4\}$ such that $P_1^{0\prime} = L_0' \cap L_1'$, $P_2^{0\prime} = L_1' \cap L_2'$, $P_3^{0\prime} = L_2' \cap L_3'$, $P_0^{0\prime} \in L_0'$, $\overline{P_0^{0\prime} P_3^{0\prime}} \subset S'$, $P_4^{0\prime} \in L_3'$, $\overline{P_1^{0\prime} P_4^{0\prime}} \subset S'$, $\overline{P_0^{1\prime} P_2^{1\prime}} \subset S'$, $\overline{P_0^{1\prime} P_3^{1\prime}} \subset S'$ and $\overline{P_1^{1\prime} P_4^{1\prime}} \subset S'$.



Since $P_0^0, P_1^0, P_2^0, P_3^0, P_4^0$ and P_0^4 (resp. $P_0^{0\prime}, P_1^{0\prime}, P_2^{0\prime}, P_3^{0\prime}, P_4^{0\prime}$) and $P_0^{4\prime}$ are in general position, there exists a linear map $\sigma: \mathbf{P}^4 \to \mathbf{P}^4$ such that $\sigma(P_0^4) = P_0^{4\prime}$ and $\sigma(P_i^0) = P_i^{0\prime}$ for all $i = 0, 1, 2, 3, 4$. Therefore, $\sigma(P_j^i) = P_j^{i\prime}$ for all i, j with $0 \leqq i, j, i+j \leqq 4$. Since $\sigma(L_0) = L_0'$, $\sigma(L_1) = L_1'$, $\sigma(L_2) = L_2'$, $\sigma(\overline{P_0^0 P_3^0}) = \overline{P_0^{0\prime} P_3^{0\prime}}$ and $\sigma(\overline{P_0^1 P_2^1}) = \overline{P_0^{1\prime} P_2^{1\prime}}$, $\sigma(S \cap H(L_0, L_2)) \cap (S' \cap H(L_0', L_2'))$ contains 5 sidtinct lines where $H(L_0, L_2)$ (resp. $H(L_0', L_2')$) is the hyperplane spanned by L_0 and L_2 (resp. L_0' and L_2'). Since degree $\sigma(S \cap H(L_0, L_2)) =$ degree $S' \cap H(L_0', L_2') = 2$, this shows that

$$\sigma(S \cap H(L_0, L_2)) = S' \cap H(L_0', L_2').$$

Similarly we have

$$\sigma(S \cap H(L_0, L_3)) = S' \cap H(L_0', L_3') \quad \text{and}$$

$$\sigma(S \cap H(L_1, L_3)) = S' \cap H(L_1', L_3').$$

Since degree $\sigma(S) =$ degree $S' = 2$ and since $\sigma(S) \cap S'$ contains three distinct quadric surfaces, we have $\sigma(S)=S'$. q.e.d.

Corollary 6.5. *$X_q(S)$ is a complete non-singular variety of dimension* 3.

Proof. Since S is complete variety, so is $X_q(S)$. For any general hyperplane H of $\mathbf{P}^4$, $S \cap H$ is non-singular quadric surface. Hence, $\dim X_q(S) \cap \Omega_{2,3}(H)=1$. Since $\operatorname{codim} \Omega_{2,3}(H)=2$, we have $\dim X_q(S)=3$ by virtue of Lemma 1.1. For any two points x and y of $X_q(S)$, there exists a biregular map σ; $\mathrm{Gr}(4, 1) \to \mathrm{Gr}(4, 1)$ such that $\sigma(X_q(S))=X_q(S)$ and $\sigma(x)=y$, by virtue of Lemma 6.3 (vi) and Lemma 6.4. This shows that $X_q(S)$ is non-singular and that every irreducible component has same dimension. Assume that $X_q(S)$ is reducible, and let X_1 and X_2 be two distinct irreducible components of $X_q(S)$. Since $X_q(S) \sim m\omega_{2,1}$ for some suitable m as a cycle of $\mathrm{Gr}(4, 1)$, we have $X_1 \sim m_1\omega_{2,1}$ and $X_2 \sim m_2\omega_{2,1}$ where m_1 and m_2 are some suitable positive integers. Since $(X_1 \cdot X_2)=m_1 \cdot m_2>0$, we have $X_1 \cap X_2 \neq \phi$. But this contradict the fact that $X_q(S)$ is non-singular. Therefore, $X_q(S)$ is a complete non-singular variety of dimension 3. q.e.d.

Lemma 6.6. *Let S be a non-singular quadric hypersurface of* $\mathbf{P}^4$. *Then,* $(X_q(S) \cdot \omega_{2,1})=4$.

Proof. Let S_0 be the non-singular quadric hypersurface of $\mathbf{P}^4$ defined by the homogeneous equation

$$X_0X_2+X_1X_3+X_4^2=0.$$

Let A_1 be the hyperplane of $\mathbf{P}^4$ defined by $X_4=0$ and let A_0 be the line of $\mathbf{P}^4$ which passes through two points $(1, 0, 0, 0, 0)$ and $(0, 0, 1, 0, 0)$. And let $x_1=\begin{pmatrix}1, 0, 0, 0, 0\\ 0, 1, 0, 0, 0\end{pmatrix}$, $x_2=\begin{pmatrix}1, 0, 0, 0, 0\\ 0, 0, 0, 1, 0\end{pmatrix}$, $x_3=\begin{pmatrix}0, 1, 0, 0, 0\\ 0, 0, 1, 0, 0\end{pmatrix}$ and $x_4=\begin{pmatrix}0, 0, 1, 0, 0\\ 0, 0, 0, 1, 0\end{pmatrix}$ be the four points of $\mathrm{Gr}(4, 1)$. Then, we have

$$X_q(S_0) \cap \Omega_{1,3}(A_0, A_1)=\{x_1, x_2, x_3, x_4\}.$$

Hence, there exists four positive integers c_1, c_2, c_3 and c_4 such that

$$X_q(S_0)\cdot \Omega_{1,3} \sim c_1x_1+c_2x_2+c_3x_3+c_4x_4\,.$$

Now we shall show that $c_1=c_2=c_3=c_4=1$. Set

$$U_{0,1}=\left\{\begin{pmatrix}1, & 0, & u_2, & u_3, & u_4\\ 0, & 1, & v_2, & v_3, & v_4\end{pmatrix} \in \mathrm{Gr}(4, 1)\right\}$$

be the affine open set of $\mathrm{Gr}(4, 1)$. $U_{0,1}$ is biregular to

$$\mathbf{A}^6=\{(u_2, u_3, u_4, v_2, v_3, v_4)\}\,.$$

Then, the defining ideal of $X_q(S_0)\cap U_{0,1}$ in $U_{0,1}$ is

$$(u_2+u_4^2,\ v_3+v_4^2,\ v_2+u_3+2u_4v_4)R$$

where $R=k[u_2, u_3, u_4, v_2, v_3, v_4]$. And the defining ideal of $\Omega_{1,3}(A_0, A_1)\cap U_{0,1}$ in $U_{0,1}$ is

$$(u_3, u_4, v_4)R\,.$$

Therefore, it is easy to see that $c_1=1$. Similarly we have $c_2=c_3=c_4=1$. Since $\Omega_{1,3}=\omega_{2,1}$, this shows that

$$(X_q(S_0)\cdot\omega_{2,1})=4\,.$$

Since $X_q(S_0)$ and $X_q(S)$ are projectively equivalent to each other,

$$(X_q(S)\cdot\omega_{2,1})=4\,. \qquad \text{q.e.d.}$$

Lemma 6.7. *Let S be a non-singular quadric hypersurface of $\mathbf{P}^4$. Denote by H_x the subset $\{y\in X_q(S) | L_x\cap L_y\neq\phi\}$ of $X_q(S)$ for any point x of $X_q(S)$. Then, we have*

(i) *H_x is an ample divisor.*

(ii) *H_x and H_y are linearly equivalent to each other for any two points x and y of $X_q(S)$.*

(iii) $H_x^3=1$.

(iv) $\dim H^0(X_q(S), \mathcal{O}(H_x)) \geqq 4$.

Proof. It is easy to see that H_x is a divisor. Fix two points x and y of $X_q(S)$ such that $L_x \cap L_y \neq \phi$. Now we shall show that

$$X_q(S) \cdot \omega_{1,0} \sim H_x + H_y .$$

Set $x = x_{1,1}$ and $y = x_{1,2}$. We can choose four points $x_{2,1}$, $x_{2,2}$, $x_{3,1}$ and $x_{3,2}$ of $X_q(S)$ usch that $L_{x_{2,1}} \cap L_{x_{2,2}} \neq \phi$, $L_{x_{3,1}} \cap L_{x_{3,2}} \neq \phi$ and $L_{x_{1,i_1}}$, $L_{x_{2,i_2}}$ and $L_{x_{3,i_3}}$ span $\mathbf{P}^4$ for all $1 \leqq i_1, i_2, i_3 \leqq 2$. Since

$$X_q(S) \cap \Omega_{2,4}(\overline{L_{x_{j,1}} L_{x_{j,2}}}) = \{H_{x_{j,1}}, H_{x_{j,2}}\} \qquad \text{for all} \quad j = 1, 2, 3$$

where $\overline{L_{x_{j,1}} L_{x_{j,2}}}$ is the linear space of dimension 2 which is spanned by $L_{x_{j,1}}$ and $L_{x_{j,2}}$, we have

$$X_q(S) \cdot \Omega_{2,4}(\overline{L_{x_{j,1}} L_{x_{j,2}}}) \sim c_{j,1} H_{x_{j,1}} + c_{j,2} H_{x_{j,2}}$$

where $c_{j,1}$ and $c_{j,2}$ are positive integers. Since $\Omega_{2,4} = \omega_{1,0}$, $\omega_{1,0}^3 = 2\omega_{2,1} + \omega_{3,0}$ and $(X_q(S) \cdot \omega_{3,0}) = 0$, we have

$$\begin{aligned}
(1) \qquad 8 &= (X_q(S) \cdot 2\omega_{2,1}) = (X_q(S) \cdot \omega_{1,0}^3) \\
&= (c_{1,1} H_{x_{1,1}} + c_{1,2} H_{x_{1,2}}) \cdot (c_{2,1} H_{x_{2,1}} + c_{2,2} H_{x_{2,2}}) \cdot \\
&\quad \cdot (c_{3,1} H_{x_{3,1}} + c_{3,2} H_{x_{3,2}}) \quad \text{in} \quad X_q(S) \\
&= \sum_{1 \leqq i_1, i_2, i_3 \leqq 2} c_{1,i_1} c_{2,i_2} c_{3,i_3} H_{x_{1,i_1}} H_{x_{2,i_2}} H_{x_{3,i_3}}
\end{aligned}$$

Since $L_{x_{1,i_1}}$, $L_{x_{2,i_2}}$ and $L_{x_{3,i_3}}$ span $\mathbf{P}^4$, we have

$$H_{x_{1,i_1}} \cdot H_{x_{2,i_2}} \cdot H_{x_{3,i_3}} \geqq 1$$

by virtue of Lemma 6.3 (v). This and the formula (1) show that

$$c_{1,1} = c_{1,2} = 1 \quad \text{and}$$

$$(2) \qquad H_{x_{1,1}} \cdot H_{x_{2,1}} \cdot H_{x_{3,1}} = 1 .$$

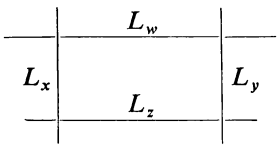
Now we shall show (ii). For any points x and y of $X_q(S)$, there exists another point z of $X_q(S)$ such that $L_x \cap L_y \neq \phi$ and $L_y \cap L_z \neq \phi$. Hence, we have

$$H_x + H_z \sim X_q(S) \cdot \omega_{1,0} \sim H_y + H_z .$$

Therefore, we have that H_x and H_y are linearly equivalent. (i) and (iii) follow easily from (ii) and (2) and the fact that $\omega_{1,0}$ is an ample divisor of $G_r(4, 1)$.

Next we shall show (iv).

We can choose four points x, y, z and w of $X_q(S)$ such that $L_x \cap L_y = \phi$, $L_z \cap L_w = \phi$, $L_x \cap L_z \neq \phi$, $L_x \cap L_w \neq \phi$, $L_y \cap L_z \neq \phi$ and $L_y \cap L_w \neq \phi$. Then, $H_x \cap H_y \cap H_z \cap H_w = \phi$. Since H_x is an ample divisor, This shows that



$\dim H^0(X_q(S), \mathcal{O}(H_x)) \geqq 4.$ q.e.d.

Lemma 6.8. *Let X be a complete non-singular variety of dimension n. Assume that there exists an ample divisor D such that $D^n = 1$ and $\dim H^0(X, \mathcal{O}(D)) \geqq n+1$. Then, X is biregular to* $\mathbf{P}^n$ (cf. *R. Goren* [2] *Theorem* 1).

Theorem 6.1 is proved by virtue of Lemma 6.4 (ii), Lemma 6.7 and Lemma 6.8.

Proof of Theorem 6.2. Let X be a subvariety of $\mathrm{Gr}(4, 1)$ which is biregular to $\mathbf{P}^3$. Assume that $(h, b, 4)$ is the triple associated with X. Then, by virtue of Theorem 4.1, Lemma 5.2 and Lemma 5.6, we have

(i) $b=0$ (in this case X is projectively equivalent to $X^0_{4,1}$) or

(ii) $(h, b)=(2, 1)$ (in this case X is projectively equivalent to $X^1_{4,1}$) or

(iii) $(h, b)=(2, 2)$.

Then, we need only to show that X is projectively equivalent to $X_q(S)$ if $(h, b)=(2, 2)$.

Let us consider the following diagram.

$$\begin{tikzcd}[column sep=small] & \mathrm{Dr}(4, 1, 0) \arrow[dl, "pr_2"'] \arrow[dr, "pr_1"] & \\ \mathrm{Gr}(4, 1) \supset X & & \mathbf{P}^4 \supset S = pr_2 \cdot pr_1^{-1}(X) \end{tikzcd}$$

where $S = pr_2 \circ pr_1^{-1}(X)$. Then, S is an irreducible variety. Since $X \cdot \Omega_{0,4} = X \cdot \omega_{3,0} = X \cdot (\omega_{1,0}^3 - 2\omega_{1,0}\omega_{1,1}) = 0$, there exists a point P of $\mathbf{P}^4$ such that $L_x \not\ni P$, for any element x of X. This shows that $S \neq \mathbf{P}^4$. It is easy to show that $\dim S \geqq 3$. Therefore, S is a hypersurface. In order to complete the proof, it is sufficient to prove the following lemma.

Lemma 6.9. *Under the same notation as above, S is a non-singular quadric hypersurface.*

Proof. Let d be the degree of S. Since $X \cdot \Omega_{1,3} = X \cdot \omega_{2,1} = X \cdot \omega_{1,0} \cdot \omega_{1,1} = 4$, we have $d \leqq 4$.

Case 1. Suppose that $d = 3$ or 4. Let P be a point of S. Since $\dim \{x \in X \mid L_x \ni P\} \geqq 1$, we have

$$\dim pr_2 \circ pr_1^{-1}(X \cap pr_1 \circ pr_2^{-1}(P)) \geqq 2.$$

We denote $pr_2 \circ pr_1^{-1}(X \cap pr_1 \circ pr_2^{-1}(P))$, by C_P. C_P is finite union of cones. And for a general point P of S, $\dim C_P = 2$, and contains a linear space of dimension 2 because $X \cdot \Omega_{1,3} = 4$. Hence, for three general points P_1, P_2 and P_3 of S, there exist linear spaces of dimension 2 A_1, A_2 and A_3 such that $C_{P_i} \supset A_i \ni P_i$ for all $i = 1, 2, 3$. We may assume that $A_i \neq A_j$ if $i \neq j$ and that A_1, A_2 and A_3 span $\mathbf{P}^4$. We denote {lines contained in A_i and pass through P_i} by $\tilde{A}_i$. For any point x of X, we have $\dim pr_1 \circ pr_2^{-1}(L_x) \geqq 2$. We denote $pr_1 \circ pr_2^{-1}(L_x)$ by H_x. Since $\dim \tilde{A}_i = 1$ and X is biregular to $\mathbf{P}^3$, $\tilde{A}_i \cap H_x \neq \phi$. This shows that $L_x \cap A_i \neq \phi$ for any x and i, hence $\dim A_i \cap A_j \geqq 1$. Since A_1, A_2 and A_3 span $\mathbf{P}^4$, we have $A_3 \supset A_1 \cap A_2$. This shows that $\{x \in \mathrm{Gr}(4, 1) \mid L_x \in P$ and $L_x \cap A_1 \cap A_2 \neq \phi\} \subset X$, for any point P of S. In particular $\{x \in \mathrm{Gr}(4, 1) \mid L_x \subset A_i\}$ is contained in X, we denote this by D_i. Since $\dim D_i=2$ and $\dim D_1 \cap D_2=0$, X is not biregular to $\mathbf{P}^3$. Therefore, $d\neq 3$ and $d\neq 4$.

Case 2. Suppose that $d=1$. Then X can be regarded as a subvariety of $\mathrm{Gr}(3, 1)$. But this is impossible by virtue of Lemma 4.7.

Case 3. Suppose that $d=2$ and S has a singular point. Let $W=\{w\in \mathrm{Gr}(4, 2)|L_w\subset S\}$ and let $D_w=\{x\in \mathrm{Gr}(4, 1)|L_x\subset L_w\}$ for any point w of W. Then, it is easy to see that

(a) $\dim W=1$ and $\dim D_w=2$.

(b) $\dim \bigcup_{w\in W} D_w=3$.

(c) $\bigcup_{w\in W} D_w\supset X$.

These show that $\#\{w\in W|D_w\subset X\}=\infty$. Hence, there exists two points w_1 and w_2, such that $D_{w_1}\cup D_{w_2}\subset X$. Since $\dim D_{w_i}=2$ and $\dim D_{w_1}\cap D_{w_2}=0$, X is not biregular to $\mathbf{P}^3$.

Therefore, S must be non-singular quadric hypersurface. q.e.d.

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