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Truncated Gamow Functions, α-Decay and the Exponential Law

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Abstract. For a quantum mechanical two-body s-wave resonance we prove that the evolution of square integrable approximations of the Gamow function is outgoing and exponentially damped. An error estimate is given in terms of resonance energy and width and the time variable. Furthermore the energy distribution is given explicitly. We obtain the Breit-Wigner form. The results are used in an α-decay model to prove general validity of the exponential decay law for periods of several lifetimes.

1. Introduction

The success of quantum mechanics to describe α-decay from a heavy nucleus is well-known. It was established by Gamow [3] and Gurney and Condon [4]. To simplify the problem the discussion is conveniently based on the following one-dimensional model: An α-particle is considered in a spherically symmetric potential $V=V(r)$ comprising a short-range negative piece from the attraction between nucleons and a positive piece of longer range from the Coulomb repulsion between protons. The large barrier thus defined by V has the effect that it confines the α-particle for a long period until it eventually escapes by tunneling. The relatively small energy differences of α-particles escaping from RaA, RaC', and Ur (examples taken from [3, 4]) account in this model for the extremely large decay rate differences observed.

Although the decay rate formulas obtained in [3, 4] are identical, the derivations are different. The idea in [3], that the α-particle is associated with a complex energy $E-i\Gamma/2$ $(=k_0^2,\ k_0=\alpha-i\beta)$, an exponentially increasing purely outgoing ($\simeq e^{ik_0r}$ for r large) space function $f(k_0,r)$ and consequently (?) an outgoing exponentially damped state, is missing in [4] and, it seems, in modern textbook derivations. E and Γ^{-1} are the energy and the lifetime, respectively.

The purpose of this paper is to put the above idea on a rigorous footing, first of all to prove that the evolution of some square integrable approximations of $f(k_0,r)$ is in fact outgoing and exponentially damped. To do this we most conveniently assume that for some $R_s>0$, $V(r)=0$ for all $r>R_s$. It is remarked that this assumption is as realistic as a repulsive Coulomb piece extending to infinity. This is due to the electrons surrounding the radioactive nucleus. Let $f_R=f(k_0,\cdot)\chi_{(0,R)}$. Then the square integrable approximations, which we investigate in detail, are given by f_{R_1} for $R_1\geqq R_s$. Letting $H=-\frac{d^2}{dr^2}+V$ on $L^2(\mathbb{R}^+)$, our main result (stated here in an imprecise way) is as follows:

For a period of many lifetimes (measured in $t\Gamma$) we have (measured by the L^2-norm)

$$e^{-itH}f_{R_1}\simeq e^{-itk_0^2}f_{R_2(t)}, \tag{1.1}$$

where $R_2(t)=2\alpha t+R_1$.

In our units the reduced mass μ of the system comprising nucleus and α-particle is equal to $1/2$ $\left(\text{the substitution } t\to\frac{t}{2\mu} \text{ introduces } \mu \text{ and real time}\right)$. Hence $R_2(t)=V_{\mathrm{cl}}\cdot t+R_1$, where V_{cl} is the classical speed corresponding to momentum α. We also remark that $\alpha\gg\beta$, and hence $E\simeq\alpha^2$.

A number of conclusions can be drawn from (1.1): First of all $e^{-itH}f_{R_1}$ is outgoing and exponentially damped. Secondly all cutoffs f_{R_1} correspond to different stages of the evolution of the same state, and hence the choice of cutoff radius R_1 is in some sense not essential. Furthermore the exponentially increasing property of $f(k_0,r)$ gets the obvious interpretation that larger distance to the barrier corresponds to earlier escape and hence, because of the decreasing outgoing flux from the barrier, we find the larger position probability there. Beyond the (free) classical propagation radius $R_2(t)$ the position probability is equal to zero. The above interpretation was already noted by Gamow to explain the (probably) unwelcome feature of $f(k_0,r)$, the infinite growth at infinity. Another consequence of (1.1) is the exponential decay law.

According to physical intuition, tunneling [because of the $\ell(\ell+1)/r^2$-barrier] is much slower for angular momentum quantum numbers $\ell\geqq 1$ than for $\ell=0$. This was already noted by Gamow and is the main reason why we restrict the entire discussion to s-wave resonances.

The present model concerns the description of α-decay by means of a simplified potential V. The α-particle state is assumed to be a truncated Gamow function. In spite of the crude simplification it seems that many phenomena can be explained within the model, qualitatively as well as quantitatively:

1. Discreteness of detected α-particle energies accompanied by some uncertainty (widths), probably (?) Breit-Wigner distributions.

2. Purely outgoing behaviour, time-delay and the decay law (observed to be very accurate for long times).

3. The connection between decay rates and energy levels (the decay rate formula).

The mathematical results of this paper are stated and proved in Sect. 3. We emphasize Lemma 3.2: The energy distribution of the state f_{R_1} is expressed in terms of the S-matrix element $S(k)$. The Breit-Wigner form is obtained. Furthermore technically Lemma 3.2 plays a central role in this paper. Our main result, corresponding to (1.1), is Theorem 3.6.

In Sect. 4 we quantitatively discuss applications to the α-decay problem: The decay law is proved in general to be valid for periods of several lifetimes. Examples from [3, 4] are numerically treated. Furthermore the propagation radius of the state $e^{-itk_0^2} f_{R_2(t)}$ is found to be $2\alpha/\Gamma$ less that the (free) classical propagation radius, see (4.2). Because 2α is the classical outgoing speed and $1/\Gamma$ is the lifetime, we arrive at the validity, in the sense of mean value, of the physical picture that the propagation is delayed corresponding to nucleus confinement for a lifetime before escape.

In the present decay model the α-particle is described by means of an intrinsic resonance state (the truncated Gamow function). We do not ask how to prepare this state. This point of view is justified by the present long lifetimes. In scattering experiments time-delay is often a phenomenon that can hardly be measured. In this case we cannot maintain the picture of a pure resonance phenomenon but complicated interference between incoming and scattered waves is present. As noted by many authors, one can "extract a resonance term" from a matrix element $(g, e^{-itH}g)$ corresponding to a scattering state g by a spectral deformation:

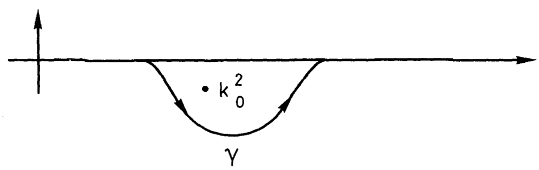

Diagram 1

As usual we find that $(g, e^{-itH}g) = Ce^{-itk_0^2} + \int_\gamma e^{-itz} \frac{d}{dz} \|P_z g\|^2 dz$ (a simple pole is assumed). An important problem in scattering theory is to find states g such that the resonance term $Ce^{-itk_0^2}$ "dominates" the second term for some time-interval. It is interesting to note that for some $g \simeq f_{R_1}$ our results provide the statement: The first term "dominates" the second for a "long" time, see Remark 3.8. The statement of Corollary 3.7 is analogous.

The physical discussion of this paper concerns α-decay, however, we believe in further applications. We refer to [1]. Various authors mention here as an open general problem to find rigorous results on the connection between lifetimes and poles of the S-matrix, on the validity of the decay law, etc. Our results are relevant also from the point of view of [6] (and authors referred to therein), where detailed knowledge of the energy distributions of resonance states in some rearrangement scattering experiments are essential for the purpose of doing numerical calculations.

In [9] we generalize our results in two directions. We prove results for all angular momentum quantum numbers, and secondly in [9] $V = V(r)$ is assumed to be a rather general, radial, and short-range potential. Explicitly, of the form of an "exterior analytic" plus an exponentially decaying potential. The result (1.1) (modified in an obvious way for $l \geqq 1$) remains valid. An explicit error estimate is given.

There exists an elementary and short proof of (1.1), which is totally different from the one presented here. The idea is the following: We shall prove that $e^{it(H-k_0^2)}f_{R_2(t)}$ is almost constant for a long period. First approximate $f_{R_2(t)}$ by a suitable smooth function $g_{R_2(t)}$. Then one realizes by performing the differentiation that $\frac{d}{dt}\{e^{it(H-k_0^2)}g_{R_2(t)}\}$ is "very small." Now (1.1) follows by integration. In this way it is possible to obtain an error estimate almost as strong as Theorem 3.6. Roughly the forms of the error constant are identical. The form is $K=C(t\Gamma)^{1/2}\left(\frac{\beta}{\alpha}\right)^{1/2}$. Due to the very explicit nature of the proof of this paper it is tempting to claim that this "form" of the constant in Theorem 3.6 is "optimal." At least it is easy to see (by the same method of proof) that as t goes to infinity, $e^{-itH}f_{R_1}$ and $e^{-itk_0^2}f_{R_2(t)}$ tend to become mutually orthogonal, so that (1.1) does not hold for t large.

The proof briefly described above has the advantage that it can be generalized rather easily to handle resonances of multiplicative non-radial potentials. This is done in [10]. Provided R_1 is "large," a result like (1.1) holds true and an explicit error estimate can be given. Unfortunately it is not possible (contrary to [9]) to control how large R_1 has to be in the non-radial case.

2. Definitions and Assumptions on V

We consider a multiplicative, radial, and real potential $V=V(r)$ satisfying the following two conditions: For some $R_s>0$, $V(r)=0$ for all $r>R_s$, and $\int_0^{R_s} r|V(r)|dr<\infty$. Let $H_0=-\frac{d^2}{dr^2}$ be the free Hamiltonian on $L^2(\mathbb{R}^+)$ determined by the boundary condition $g(0)=0$ for $g\in D(H_0)$. Then it is easy to prove that V is infinitesimally form-bounded with respect to H_0. Hence we can construct the total Hamiltonian $H=H_0+V$ by the standard quadratic form technique.

The following functions (and notation) can all be found in Newton [7, Sect. 12.1].

We consider for $k\in\mathbb{R}\setminus\{0\}$ solutions $\varphi(k,r)$, $f(k,r)$, and $\psi^+(k,r)$ of the equation $\left(-\frac{d^2}{dr^2}+V(r)-k^2\right)\psi(r)=0$. $\varphi(k,r)$ is given by the conditions $\varphi(k,0)=0$ and $\frac{d}{dr}\varphi(k,0)=1$, and $f(k,r)$ by the condition $f(k,r)=e^{ikr}$ for $r\geqq R_s$. $\psi^+(k,r)$ is equal to $\frac{k\varphi(k,r)}{F(k)}$, where the Jost function $F(k)$ is given by $F(k)=W(f(k,r),\varphi(k,r))=\frac{d}{dr}\varphi(k,r)f(k,r)-\varphi(k,r)\frac{d}{dr}f(k,r)$, the Wronskian between $\varphi(k,r)$ and $f(k,r)$. It is known that $F(k)\neq 0$.

The connection between $\varphi(k,r)$ and $f(\pm k,r)$ is given by

$$\varphi(k,r)=\frac{1}{2ik}(F(-k)f(k,r)-F(k)f(-k,r)).\tag{2.1}$$

Using $\overline{F(k)}=F(-k)$ the "unitary property" of

$$S(k)=\frac{F(-k)}{F(k)}, \quad |S(k)|=1, \quad \text{follows}. \tag{2.2}$$

Because $\varphi(k,r)$ is real, also the equation

$$\overline{\psi^+(k,r)}=\frac{k\varphi(k,r)}{F(-k)} \quad \text{follows}. \tag{2.3}$$

In this paper we make use of the following expression for the kernel of the spectral density of H, $\frac{dE_\lambda}{d\lambda}$, where we put $\lambda=k^2$ and $k>0$.

$$\frac{dE_\lambda}{d\lambda}(r,r')=\frac{1}{k\pi}\,\psi^+(k,r)\,\overline{\psi^+(k,r')}.$$

The expression is found utilizing the identity $2\pi i\frac{dE_\lambda}{d\lambda}=\lim_{\varepsilon\downarrow 0}\{(H-\lambda-i\varepsilon)^{-1} -(H-\lambda+i\varepsilon)^{-1}\}$, the fact that the kernel of $(H-\lambda^2\mp i0)^{-1}$ is given (cf. [7, (12.40)]) by

$$\frac{\varphi(k,r)f(\pm k,r)}{F(\pm k)} \quad \text{for} \quad r\leqq r' \quad \text{and} \quad \frac{\varphi(k,r')f(\pm k,r)}{F(\pm k)} \quad \text{for} \quad r>r',$$

and the formulas (2.1), (2.3).

The orthogonal projections onto the subspace of absolute continuity of H and the span of all eigenvectors of H are given by $P_{ac}:=I-E_0$ and $P_e:=E_0 =\chi_{(-\infty,0]}(H)$, respectively.

The functions $\varphi(k,r)$, $f(k,r)$, and $F(k)$ admit analytic extension in k to the whole k-plane. A resonance is defined to be a point $k_0=\alpha-i\beta$ $(\alpha,\beta>0)$, where $F(k_0) =f(k_0,0)=0$. We define the resonance energy E and the width Γ by $k_0^2=\alpha^2-\beta^2 -i2\alpha\beta=E-i\Gamma/2$. For $R\geqq R_s$ we introduce truncated Gamow functions f_R given by $f_R=f(k_0,\cdot)\chi_{(0,R)}$.

Using that $\overline{f(k_0,r)}$ is solution of $\left(-\frac{d^2}{dr^2}+V(r)-\overline{k_0^2}\right)\psi(r)=0$ we find,

$$\frac{d}{dr}W(\overline{f(k_0,r)},f(k_0,r))=(\overline{k_0^2}-k_0^2)|f(k_0,r)|^2. \tag{2.4}$$

Similarly for $k\in\mathbb{R}^+$,

$$\frac{d}{dr}W(\varphi(k,r),f(k_0,r))=(k^2-k_0^2)\varphi(k,r)f(k_0,r). \tag{2.5}$$

3. The Mathematical Results

Throughout this section a resonance k_0 is fixed. In the Lemmas 3.1–3.3 to be given below we fix $R\geqq R_s$.

The formula in Lemma 3.1 is well-known. It was applied for instance by Ashbaugh and Harrell [2, 5] to express β in terms of $f(k_0, r)$. However, for our purpose the formula is useful of the form where $\|f_R\|$ is given in terms of β.

Lemma 3.1. $\|f_R\|^2 = \int_0^R |f(k_0, r)|^2 dr = \frac{e^{2\beta R}}{2\beta}.$

Proof. We integrate (2.4) and use the known boundary properties of $f(k_0, r)$ at $r = R$ together with the resonance condition $f(k_0, 0) = 0$.

Lemma 3.2. *For* $k > 0$,

$$\langle \psi^+(k, \cdot), f_R \rangle = \int_0^\infty \overline{\psi^+(k, r)} f_R(r) dr$$
$$= \frac{-k}{k^2 - k_0^2} S(-k) e^{i(k_0 - k)R} \left[1 + \frac{k - k_0}{2k} (S(k) e^{2ikR} - 1) \right].$$

Proof. By integrating (2.5) and using the known boundary conditions, we obtain

$$\int_0^R \varphi(k, r) f(k_0, r) dr = \frac{-1}{k^2 - k_0^2} \Big(-\varphi(k, R) i k_0 e^{ik_0 R} + \frac{d}{dr} \varphi(k, R) e^{ik_0 R} \Big). \tag{3.1}$$

We apply (2.1) in the following calculation:

$$\left(-\varphi(k, R) i k_0 e^{ik_0 R} + \frac{d}{dr} \varphi(k, R) e^{ik_0 R} \right)$$
$$= e^{i(k_0 - k)R} \left(-\varphi(k, R) i k e^{ikR} + \frac{d}{dr} \varphi(k, R) e^{ikR} + \varphi(k, R) e^{ikR} i(k - k_0) \right)$$
$$= e^{i(k_0 - k)R} F(k) \left(1 + i(k - k_0) e^{ikR} \frac{\varphi(k, R)}{F(k)} \right)$$
$$= e^{i(k_0 - k)R} F(k) \left(1 + \frac{(k - k_0)}{2k} e^{ikR} [S(k) f(k, R) - f(-k, r)] \right)$$
$$= e^{i(k_0 - k)R} F(k) \left(1 + \frac{(k - k_0)}{2k} [S(k) e^{2ikR} - 1] \right). \tag{3.2}$$

Now the lemma easily follows from (2.3), (3.1), and (3.2).

Lemma 3.3. *The following estimate holds true.*

$$\|P_e(f_R)\|^2 \leqq \frac{2}{\pi} \left(\pi^{1/2} \left(\frac{\beta}{\alpha} \right)^{1/2} + \frac{\beta}{\alpha} \right) \|f_R\|^2.$$

Proof. Using $\|P_{ac}(f_R)\|^2 = \frac{2}{\pi} \int_0^\infty |\langle \psi^+(k, \cdot), f_R \rangle|^2 dk$, (2.2) and Lemma 3.2 we find

$$\|P_e(f_R)\|^2 = \|f_R\|^2 - \|P_{ac}(f_R)\|^2 = \|f_R\|^2 - \frac{2}{\pi} \int_0^\infty dk \frac{k^2}{|k^2 - k_0^2|^2} e^{2\beta R} \{a + b + c\}, \tag{3.3}$$

where

$$a = 1, \quad b = -\mathrm{Re}\left\{\frac{k-k_0}{k}\right\} + \frac{1}{2}\left|\frac{k-k_0}{k}\right|^2,$$

and

$$c = \mathrm{Re}\left\{\frac{k-k_0}{k} S(k)e^{2ikR} - \left|\frac{k-k_0}{2k}\right|^2 2S(k)e^{2ikR}\right\}.$$

We now prove that the right-hand side of (3.3) is equal to $-\frac{2}{\pi}\int_0^\infty dk \frac{k^2}{|k^2-k_0^2|^2} e^{2\beta R} c$ (i.e. the other terms cancel).

The following two integrals, calculated by changing the contour of integration $(0, \infty) \to (0, -\infty)$ and using Cauchy's theorem, are useful:

$$\int_0^\infty dk \frac{k^2}{|k^2-k_0^2|^2} = \frac{\pi}{4\beta}, \tag{3.4}$$

$$\int_0^\infty dk \frac{1}{|k^2-k_0^2|^2} = \frac{1}{|k_0^2|}\frac{\pi}{4\beta}. \tag{3.5}$$

According to Lemma 3.1 and (3.4),

$$\|f_R\|^2 = \frac{2}{\pi} e^{2\beta R} \int_0^\infty \frac{k^2}{|k^2-k_0^2|^2} dk. \tag{3.6}$$

It follows from (3.3) and (3.6) that

$$\|P_e(f_R)\|^2 = -\frac{2}{\pi} e^{2\beta R} \int_0^\infty dk \frac{k^2}{|k^2-k_0^2|^2}\{b+c\}.$$

We remark that $b = \frac{1}{2}\left(\frac{|k_0|^2}{k^2} - 1\right)$, and thus (3.4) and (3.5) imply that

$$\int_0^\infty dk \frac{k^2}{|k^2-k_0^2|^2} b = \frac{1}{2}|k_0|^2 \int_0^\infty dk \frac{1}{|k^2-k_0^2|^2} - \frac{1}{2}\int_0^\infty dk \frac{k^2}{|k^2-k_0^2|^2} = 0.$$

The identity

$$\|P_e(f_R)\|^2 = -\frac{2}{\pi}\int_0^\infty dk \frac{k^2}{|k^2-k_0^2|^2} e^{2\beta R} c \tag{3.7}$$

is proved.

To complete the proof of the lemma the following estimates are useful.

$$\int_0^\infty dk \frac{1}{|k+k_0|^2} < \alpha^{-1}, \tag{3.8}$$

and

$$\int_0^\infty dk \frac{k^2}{|k^2-k_0^2|^2}\left|\frac{k-k_0}{k}\right| \leqq \left(\int_0^\infty dk \frac{k^2}{|k^2-k_0^2|^2}\right)^{1/2}\left(\int_0^\infty dk \frac{1}{|k+k_0|^2}\right)^{1/2}$$
$$< \frac{1}{2}\left(\frac{\pi}{\beta}\right)^{1/2} \alpha^{-1/2}. \tag{3.9}$$

In the last step of (3.9) we have used (3.4) and (3.8).

From (3.7), the estimate $|c| \leqq \left|\frac{k-k_0}{k}\right| + \frac{1}{2}\left|\frac{k-k_0}{k}\right|^2$, (3.8) and (3.9), we immediately get that

$$\|P_e(f_R)\|^2 < \frac{2}{\pi} e^{2\beta R}\left(\frac{1}{2}\left(\frac{\pi}{\beta}\right)^{1/2} \alpha^{-1/2} + \frac{1}{2}\alpha^{-1}\right) = \frac{e^{2\beta R}}{2\beta}\frac{2}{\pi}\left(\pi^{1/2}\left(\frac{\beta}{\alpha}\right)^{1/2} + \frac{\beta}{\alpha}\right).$$

Hence, taking also Lemma 3.1 into account, the lemma is proved.

Remark 3.4. The bound in Lemma 3.3 does not depend on R. However (as expected), it is true that $\|P_e(f_R)\|/\|f_R\| \to 0$ for $R \to \infty$. This fact follows from (3.7) and the Riemann-Lebesgue lemma.

In the remaining part of this section we fix $R_1 \geqq R_s$.

Lemma 3.5. *Introducing $R_2 = R_2(t) = 2\alpha t + R_1$, $t \geqq 0$, we have the estimate*

$$\|P_{ac}(e^{-itH} f_{R_1} - e^{-itk_0^2} f_{R_2})\|^2$$
$$\leqq \|f_{R_1}\|^2 \frac{16}{\pi}\left\{\left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{\beta}{\alpha}\right)^{1/4}\right.$$
$$\left. \times \left\{\frac{3}{16}\pi(t\Gamma)^{1/2}\left(1+20\left(\frac{\beta}{\alpha}\right)^{1/2}\right)^2\left(1+10\left(\frac{\beta}{\alpha}\right)^{1/2}\right)^{-2} + \frac{3}{40}\right\}^{1/2}\right\}^2.$$

Proof. By Lemma 3.2,

$$\begin{aligned} x(k,t) &:= \langle \psi^+(k,\cdot), e^{-itk^2} f_{R_1}\rangle - \langle \psi^+(k,\cdot), e^{-itk_0^2} f_{R_2}\rangle \\ &= \frac{-k}{k^2-k_0^2} S(-k)\left\{e^{-itk^2} e^{i(k_0-k)R_1}\left[1+\frac{k-k_0}{2k}(S(k)e^{2ikR_1}-1)\right]\right. \\ &\quad \left. - e^{-itk_0^2} e^{i(k_0-k)R_2}\left[1+\frac{k-k_0}{2k}(S(k)e^{2ikR_2}-1)\right]\right\} \\ &= \frac{-k}{k^2-k_0^2} S(-k)\{a+b+c\}, \end{aligned}$$

where

$$a = e^{-itk^2} e^{i(k_0-k)R_1}\frac{k-k}{2k}(S(k)e^{2ikR_1}-1),$$

$$b = -e^{-itk_0^2} e^{i(k_0-k)R_2}\frac{k-k_0}{2k}(S(k)e^{2ikR_2}-1),$$

and

$$c=e^{-itk^2}e^{i(k_0-k)R_1}(1-e^{it(k^2-k_0^2)}e^{i(k_0-k)2\alpha t}).$$

Using Cauchy-Schwarz' inequality we find

$$\begin{aligned}\|P_{ac}(e^{-itH}f_{R_1}-e^{-itk_0^2}f_{R_2})\|^2 &= \frac{2}{\pi}\int_0^\infty |x(k,t)|^2 dk\\ &\leqq \frac{2}{\pi}\left(\left\{\int_0^\infty dk\frac{k^2}{|k^2-k_0^2|^2}|a|^2\right\}^{1/2}\right.\\ &\quad+\left\{\int_0^\infty dk\frac{k^2}{|k^2-k_0^2|^2}|b|^2\right\}^{1/2}\\ &\quad\left.+\left\{\int_0^\infty dk\frac{k^2}{|k^2-k_0^2|^2}|c|^2\right\}^{1/2}\right)^2. \end{aligned}\tag{3.10}$$

Because $|a|^2, |b|^2 \leqq e^{2\beta R_1}\left|\dfrac{k-k_0}{k}\right|^2$, (3.8) immediately provides the estimates

$$\int_0^\infty dk\frac{k^2}{|k^2-k_0^2|^2}|a|^2 \leqq e^{2\beta R_1}\alpha^{-1},\tag{3.11}$$

$$\int_0^\infty dk\frac{k^2}{|k^2-k_0^2|^2}|b|^2 \leqq e^{2\beta R_1}\alpha^{-1}.\tag{3.12}$$

The proof of the following estimate will be given later.

$$\begin{aligned}&\int_0^\infty dk\frac{k^2}{|k^2-k_0^2|^2}|c|^2\\ &\leqq e^{2\beta R_1}\left\{\frac{3}{2}\pi t^{1/2}\left(1+20\left(\frac{\beta}{\alpha}\right)^{1/2}\right)^2\left(1+10\left(\frac{\beta}{\alpha}\right)^{1/2}\right)^{-2}+\frac{3}{10}\alpha^{-1/2}\beta^{-1/2}\right\}.\end{aligned}\tag{3.13}$$

We insert (3.11)–(3.13) into the right-hand side of (3.10) and prove the lemma:

$$\begin{aligned}&\|P_{ac}(e^{-itH}f_{R_1}-e^{-itk_0^2}f_{R_2})\|^2\\ &\leqq \frac{2}{\pi}e^{2\beta R_1}\left\{2\alpha^{-1/2}+\left\{\frac{3}{2}\pi t^{1/2}\left(1+20\left(\frac{\beta}{\alpha}\right)^{1/2}\right)^2\right.\right.\\ &\quad\left.\left.\times\left(1+10\left(\frac{\beta}{\alpha}\right)^{1/2}\right)^{-2}+\frac{3}{10}\alpha^{-1/2}\beta^{-1/2}\right\}^{1/2}\right\}^2\\ &=\|f_{R_1}\|^2\frac{16}{\pi}\left\{\left(\frac{\beta}{\alpha}\right)^{1/2}+\left(\frac{\beta}{\alpha}\right)^{1/4}\right.\\ &\quad\left.\times\left\{\frac{3}{16}\pi(t\Gamma)^{1/2}\left(1+20\left(\frac{\beta}{\alpha}\right)^{1/2}\right)^2\left(1+10\left(\frac{\beta}{\alpha}\right)^{1/2}\right)^{-2}+\frac{3}{40}\right\}^{1/2}\right\}^2.\end{aligned}$$

Proof of (3.13). *Clearly*

$$|c|^2=e^{2\beta R_1}|1-e^{it(k^2-k_0^2)}e^{i(k_0-k)2\alpha t}|^2.$$

By inserting $k^2-k_0^2=2\alpha(k-\alpha)+(k-\alpha)^2+\beta^2+i\Gamma/2$ and $(k_0-k)2\alpha=-2\alpha(k-\alpha)-i\Gamma/2$, we obtain

$$|c|^2=e^{2\beta R_1}|1-e^{it((k-\alpha)^2+\beta^2)}|^2=e^{2\beta R_1}2(1-\cos(t|k-k_0|^2)). \tag{3.14}$$

We shall utilize the following inequality valid for all $x\geqq 0$:

$$1-\cos x\leqq 3\frac{x}{1+x}. \tag{3.15}$$

Remark that $(1+x)\frac{1-\cos x}{x}\leqq(1+x)\frac{x}{2}$ and that $\left(\frac{1}{x}+1\right)(1-\cos x)\leqq\left(\frac{1}{x}+1\right)2$.

Equations (3.14) and (3.15) provide the estimate

$$|c|^2\leqq e^{2\beta R_1}6\frac{t|k-k_0|^2}{1+t|k-k_0|^2}. \tag{3.16}$$

We now proceed as follows, using (3.16):

$$\begin{aligned}
&\int_0^\infty dk\frac{k^2}{|k^2-k_0^2|^2}|c|^2\\
&\leqq e^{2\beta R_1}6\int_0^\infty dk\frac{k^2}{|k^2-k_0^2|^2}\,\frac{t|k-k_0|^2}{1+t|k-k_0|^2}\\
&\leqq e^{2\beta R_1}6\int_0^\infty dk\frac{k^2}{|k+\alpha|^2}\,\frac{t}{1+t|k-\alpha|^2}\\
&=e^{2\beta R_1}6t^{1/2}\int_{-\alpha t^{1/2}}^{\infty}dz\left|\frac{z+t^{1/2}\alpha}{z+2t^{1/2}\alpha}\right|^2\frac{1}{1+z^2}\qquad(z=t^{1/2}(k-\alpha))\\
&=e^{2\beta R_1}6t^{1/2}\left(\int_{-\alpha t^{1/2}}^{10(t\Gamma)^{1/2}}+\int_{10(t\Gamma)^{1/2}}^{\infty}\right)\\
&\leqq e^{2\beta R_1}6t^{1/2}\left(\left|\frac{10(t\Gamma)^{1/2}+t^{1/2}\alpha}{10(t\Gamma)^{1/2}+2t^{1/2}\alpha}\right|^2\int_{-\infty}^{\infty}dz\frac{1}{1+z^2}+\int_{10(t\Gamma)^{1/2}}^{\infty}dz\frac{1}{z^2}\right)\\
&=e^{2\beta R_1}\left\{\frac{3}{2}\pi t^{1/2}\left(1+20\left(\frac{\beta}{\alpha}\right)^{1/2}\right)^2\left(1+10\left(\frac{\beta}{\alpha}\right)^{1/2}\right)^{-2}+\frac{3}{10}\alpha^{-1/2}\beta^{-1/2}\right\}.
\end{aligned}$$

The proof of (3.13), and hence the lemma, is complete.

Our main result is the following theorem.

Theorem 3.6. *Let $R_2=R_2(t)=2\alpha t+R_1$, $t\geqq 0$. Then*

$$\|e^{-itH}f_{R_1}-e^{-itk_0^2}f_{R_2}\|^2\leqq\|f_{R_1}\|^2\frac{16}{\pi}K(\alpha,\beta,t),$$

where

$$\begin{aligned}
K(\alpha,\beta,t)=\frac{1}{2}\left(\pi^{1/2}\left(\frac{\beta}{\alpha}\right)^{1/2}+\frac{\beta}{\alpha}\right)+\Bigg\{\left(\frac{\beta}{\alpha}\right)^{1/2}+\left(\frac{\beta}{\alpha}\right)^{1/4}\\
\times\left\{\frac{3}{16}\pi(t\Gamma)^{1/2}\left(1+20\left(\frac{\beta}{\alpha}\right)^{1/2}\right)^2\left(1+10\left(\frac{\beta}{\alpha}\right)^{1/2}\right)^{-2}+\frac{3}{40}\right\}^{1/2}\Bigg\}^2.
\end{aligned}$$

Proof. We use that

$$\|e^{-itH}f_{R_1}-e^{-itk_0^2}f_{R_2}\|^2=\|P_{\mathrm{ac}}(e^{-itH}f_{R_1}-e^{-itk_0^2}f_{R_2})\|^2 + \|P_{\mathrm{e}}(e^{-itH}f_{R_1}-e^{-itk_0^2}f_{R_2})\|^2 .$$

The first term is estimated as in Lemma 3.5. The second as follows:

$$\begin{aligned}&\|P_{\mathrm{e}}(e^{-itH}f_{R_1}-e^{-itk_0^2}f_{R_2})\|^2\\ &\leqq 2(\|P_{\mathrm{e}}(e^{-itH}f_{R_1})\|^2+\|P_{\mathrm{e}}(e^{-itk_0^2}f_{R_2})\|^2)\\ &\leqq 2(\|f_{R_1}\|^2+e^{-\Gamma t}\|f_{R_2}\|^2)\frac{2}{\pi}\left(\pi^{1/2}\left(\frac{\beta}{\alpha}\right)^{1/2}+\frac{\beta}{\alpha}\right) \quad \text{(Lemma 3.3)}\\ &=\|f_{R_1}\|^2\frac{8}{\pi}\left(\pi^{1/2}\left(\frac{\beta}{\alpha}\right)^{1/2}+\frac{\beta}{\alpha}\right) \quad \text{(Lemma 3.1)}.\end{aligned}$$

We have finished the proof.

Theorem 3.6 almost immediately implies the following Corollaries 3.7 and 3.10, which concern two different "measures of decay."

Corollary 3.7. *For all* $t\geqq 0$,

$$|(f_{R_1}, e^{-itH}f_{R_1})|^2\|f_{R_1}\|^{-4}=e^{-\Gamma t}|1+x(t)|^2 ,$$

where $|x(t)|\leqq e^{\Gamma t/2}4\pi^{-1/2}K(\alpha,\beta,t)^{1/2}$.

Proof.

$$\begin{aligned}(f_{R_1}, e^{-itH}f_{R_1})&=(f_{R_1}, e^{-itk_0^2}f_{R_2})+(f_{R_1}, e^{-itH}f_{R_1}-e^{-itk_0^2}f_{R_2})\\ &=e^{-itk_0^2}\|f_{R_1}\|^2\times\{1+e^{itk_0^2}\|f_{R_1}\|^{-2}(f_{R_1}, e^{-itH}f_{R_1}-e^{-itk_0^2}f_{R_2})\} .\end{aligned}$$

Cauchy-Schwarz' inequality and Theorem 3.6 now complete the proof.

Remark 3.8. There exists another proof of Corollary 3.7 based directly on Lemmas 3.1–3.3. One "extracts" the term

$$4\beta\pi^{-1}\int_0^\infty dk e^{-itk^2}\frac{k^2}{|k^2-k_0^2|^2} \quad \text{from} \quad (P_{\mathrm{ac}}f_{R_1}, e^{-itH}f_{R_1})\|f_{R_1}\|^{-2},$$

and then uses Cauchy's theorem on the integral (as in the proof of Lemma 3.3). A slight improvement of Corollary 3.7 is obtained [the factor $(t\Gamma)^{1/4}$ is avoided].

Remark 3.9. The bad behaviour of the bound in Corollary 3.7 as $t\to\infty$ is expected. In fact, it can be proved that there exists no $g\neq 0$ such that for all $t>0$, $|(g, e^{-itH}g)| <Ke^{-\varepsilon t}$, K, $\varepsilon>0$. This is well-known, see for instance Simon [8].

Corollary 3.10. *Let* $R_3\geqq R_1$ *be given and define* $D=\min\{e^{\Gamma t}, e^{2\beta(R_3-R_1)}\}$, $t\geqq 0$. *Then*

$$(e^{-itH}f_{R_1}, \chi_{(0,R_3)}e^{-itH}f_{R_1})\|f_{R_1}\|^{-2}=e^{-\Gamma t}D(1+y(t)) ,$$

where

$$|y(t)|\leqq 16\pi^{-1}D^{-1}e^{\Gamma t}K(\alpha,\beta,t)+8\pi^{-1/2}D^{-1/2}e^{\Gamma t/2}K(\alpha,\beta,t)^{1/2} .$$

Proof.

$$\begin{aligned}&(e^{-itH}f_{R_1}, \chi_{(0,R_3)}e^{-itH}f_{R_1})\\&\quad= e^{-\Gamma t}(f_{R_2}, \chi_{(0,R_3)}f_{R_2}) + (\{e^{-itH}f_{R_1} - e^{-itk_0^2}f_{R_1}\}, \chi_{(0,R_3)}\{e^{-itH}f_{R_1} - e^{-itk_0^2}f_{R_2}\})\\&\qquad+2\,\mathrm{Re}(\{e^{-itH}f_{R_1} - e^{-itk_0^2}f_{R_2}\}, \chi_{(0,R_3)}e^{-itk_0^2}f_{R_2}).\end{aligned}$$

By Lemma 3.1 the first term is equal to

$$e^{-\Gamma t}D\|f_{R_1}\|^2 \quad \text{and} \quad \|\chi_{(0,R_3)}e^{-itk_0^2}f_{R_2}\| = e^{-\Gamma t/2}D^{1/2}\|f_{R_1}\|.$$

As before we can now complete the proof using Cauchy-Schwarz' inequality and Theorem 3.6.

4. The Physical Applications

Within the framework of our simplified α-decay model we now present a proof of the validity of the exponential decay law for some time-interval. We calculate the probability P_t, that an α-particle is detected during the time-interval $(0, t)$. Letting R_3 be the radius of detection we have according to Corollary 3.10 [D and $y(t)$ given there] that

$$P_t = 1 - \|\chi_{(0,R_3)}e^{-itH}f_{R_1}\|^2\|f_{R_1}\|^{-2} = 1 - e^{-\Gamma t}D(1 + y(t)).$$

f_{R_1} ($R_1 \leqq R_3$) is assumed to be the α-particle state at the time $t = 0$. Thus for $t \geqq 2\beta(R_3 - R_1)\Gamma^{-1}$,

$$P_t = 1 - e^{-\Gamma(t - 2\beta(R_3 - R_1)\Gamma^{-1})}(1 + y(t)). \tag{4.1}$$

The "delay" term $2\beta(R_3 - R_1)\Gamma^{-1}$ is due to finiteness of the speed of the escaping α-particle. In fact, explicitly $2\beta(R_3 - R_1)\Gamma^{-1} = (R_3 - R_1)V_{\mathrm{cl}}{}^{-1}$, where $V_{\mathrm{cl}} = 2\alpha$ is the classical speed of a free α-particle with energy $E \simeq \alpha^2$. In most experiments $R_1 = R_3$ is the "right" choice of R_1.

If for some time-interval $(2\beta(R_3 - R_1)\Gamma^{-1}, t_0)$, $|y(t)|$ is "small" compared to 1, then (4.1) is precisely the law of exponential decay [an integrated exponential distribution function then has the form (4.1)]. Table 1 below indicates the lengths of time-intervals $(2\beta(R_3 - R_1)\Gamma^{-1}, t_0)$ for *RaC'*, *RaA*, and *Ur* respectively, such that $|y(t)| < 0.2$ or $|y(t)| < 0.01$. The data in the first two rows have been taken from [4]. The third row gives $2\beta R_3$ in the case $R_3 = 1m$, the fourth row Γ/E. We remark that in the evaluation of $|y(t)|$ we can use $\frac{1}{4} \cdot \Gamma/E$ instead of the quantity β/α. Also we remark that $2\beta R_3 \ll t_0\Gamma$ for physically realistic values of R_3 (the third and last row), so also in the case $R_1 < R_3$, $2\beta(R_3 - R_1)\Gamma^{-1}$ is negligible compared to t_0.

Table 1

	RaC'	*RaA*	*Ur*
Lifetime Γ^{-1}	4.4 10^{-8} mi.	4.4 mi.	4.4 10^{15} mi.
Speed	1.92 10^9 cm/s	1.69 10^9 cm/s	1.4 10^9 cm/s
$2\beta R_3$, $R_3 = 1$ m	2 10^{-2}	2 10^{-10}	3 10^{-25}
Γ/E	3 10^{-17}	4 10^{-25}	6 10^{-40}
$\lvert y(t)\rvert < 0.2$ for $t_0\Gamma =$	12	21	37
$\lvert y(t)\rvert < 0.01$ for $t_0\Gamma =$	6	15	32

It is interesting to note that the sharp cutoffs f_{R_1} are chosen only for technical convenience. For instance we could use smooth cutoff approximations of the Gamow function as well. This is due to the fact that in a part of the space defined by $r \geqq R_s$, and for instance $r < R_1 = 1$ m, $|f_{R_1}(r)|^2$ is "very small" so that

$$\int_{R_s}^{R_1} |f_{R_1}(r)|^2 dr \ll \int_0^{R_s} |f_{R_1}(r)|^2 dr \quad \text{(the third row and Lemma 3.1)}.$$

Hence it is clear that any smooth cutoff (in distance larger than R_s) approximation of the Gamow function is "very close" (measured by the L^2-norm) to some sharp cutoff approximation f_{R_1}, and thus we could use our mathematical result to handle smooth cutoff approximations also.

Justified by Theorem 3.6 it is interesting to calculate the propagation radius $(e^{-itk_0^2} f_{R_2}, re^{-itk_0^2} f_{R_2}) \|f_{R_1}\|^{-2}$ of the state $e^{-itk_0^2} f_{R_2} \cdot \|f_{R_1}\|^{-1}$. This straightforward calculation is omitted. We remark that the trivial inequalities, $0 < \int_0^{R_1} r|f_{R_2}(r)|^2 dr < R_1 \frac{e^{2\beta R_1}}{2\beta}$, and the decomposition $(f_{R_2}, rf_{R_2}) = \int_0^{R_1} + \int_{R_1}^{R_2}$ are utilized. The result is

$$(e^{-itk_0^2} f_{R_2}, re^{-itk_0^2} f_{R_2}) \|f_{R_1}\|^{-2} = R_{\mathrm{cl}} - V_{\mathrm{cl}} \Gamma^{-1}(1 - z(t)), \tag{4.2}$$

where $R_{\mathrm{cl}} = R_2 = R_1 + 2\alpha t$, $V_{\mathrm{cl}} = 2\alpha$ and

$$(1 - R_1 \Gamma / V_{\mathrm{cl}}) e^{-\Gamma t} = (1 - 2\beta R_1) e^{-\Gamma t} < z(t) < e^{-\Gamma t}.$$

We can assume $2\beta R_1 < 1$. The physical interpretation of (4.2) is obvious: The propagation is delayed corresponding to nucleus confinement for a lifetime.

Using $e^{-itk_0^2} f_{R_2}$, we can calculate the outgoing flux at distance $R < R_2$ ($R \geqq R_s$):

$$-\frac{d}{dt} \|\chi_{(0,R)} e^{-itk_0^2} f_{R_2}\|^2 \cdot \|f_{R_1}\|^{-2} = \Gamma e^{-\Gamma t} e^{2\beta(R - R_1)}. \tag{4.3}$$

Thus at fixed t the flux increases as R increases. This is physically expected. Larger R corresponds to earlier escape, and hence larger outgoing flux from the barrier [this time dependence of the flux also follows from (4.3)]. The above calculated flux is also given by

$$\lim_{\Delta t \downarrow 0} (\Delta t)^{-1} \|\chi_{(R - V_{\mathrm{cl}} \cdot \Delta t, R)} e^{-itk_0^2} f_{R_2}\|^2 \|f_{R_1}\|^{-2}.$$

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