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Yang-Mills on Surfaces with Boundary: Quantum Theory and Symplectic Limit

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Abstract: The quantum field measure for gauge fields over a compact surface with boundary, with holonomy around the boundary components specified, is constructed. Loop expectation values for general loop configurations are computed. For a compact oriented surface with one boundary component, let $\mathcal{M}(\Theta)$ be the moduli space of flat connections with boundary holonomy lying in a conjugacy class Θ in the gauge group G. We prove that a certain natural closed 2−form on $\mathcal{M}(\Theta)$, introduced in an earlier work by C. King and the author, is a symplectic structure on the generic stratum of $\mathcal{M}(\Theta)$ for generic Θ. We then prove that the quantum Yang-Mills measure, with the boundary holonomy constrained to lie in Θ, converges in a natural sense to the corresponding symplectic volume measure in the classical limit. We conclude with a detailed treatment of the case $G = SU(2)$, and determine the symplectic volume of this moduli space.

1. Introduction and Overview of Results

This paper presents the construction of a quantum gauge field measure over compact surfaces, with specified boundary holonomies, and a determination of the classical limit of this measure when the surface is oriented and has one boundary component.

Results concerning the quantum field measure. The construction of the measure and determination and study of the loop expectation values are carried out in Sects. 1–5. In these sections:

(i) We construct the Euclidean quantum field measure for gauge theory over a compact surface with boundary, with boundary holonomy (or its conjugacy class) specified (the gauge group is a compact connected Lie group).

(ii) Loop expectation values are computed explicitly, and it is shown that they are invariant under appropriate area-preserving surface homeomorphisms.

The loop expectation value formulas we obtain are quite natural in view of the free-boundary expectation value formulas available in [Fi, Se2,3, Wi1]. Thus one could take them simply as a starting point, rather than a conclusion, from the point of view of lattice gauge theory. However, our objective in this paper has been to *derive* these expectation value formulas from a continuum theory.

<u>*Results concerning the classical limit and sympectic volume*</u>. In Sects. 6–11, we focus attention on a compact oriented surface Σ of positive genus with one boundary component $\partial\Sigma$. Let $\mathcal{M}(\Theta)$ denote the moduli space of flat connections over Σ with holonomy around $\partial\Sigma$ lying in a given conjugacy class Θ in the gauge group G. The gauge group is now taken to be compact connected semisimple. In [KS2] a 2–form Ω_Θ was defined on the space $\mathcal{A}(\Theta)$ of all connections whose holonomy around $\partial\Sigma$ lie in Θ. It was shown that Ω_Θ descends naturally to a closed 2–form $\overline{\Omega}_\Theta$ on the moduli space $\mathcal{M}(\Theta) = \mathcal{A}^0(\Theta)/\mathcal{G}$, where $\mathcal{A}^0(\Theta)$ is the set of flat connections in $\mathcal{A}(\Theta)$ and $\mathcal{G}$ is the group of gauge transformations. In [KS2] it was proven that $\overline{\Omega}_\Theta$ is non-degenerate on the 'smooth part' of $\mathcal{M}(\Theta)$ when the conjugacy class Θ passes through a certain neighborhood of the identity in G; the proof of non-degeneracy in [KS2] was obtained by 'perturbation' of the case $\Theta = \{e\}$, the latter case being dealt with by means of an earlier result in [KS1]. In the present paper we prove the following results:

(iii) $\overline{\Omega}_\Theta$ <u>is non-degenerate</u> on (the 'smooth part' of) $\mathcal{M}(\Theta)$ for generic conjugacy classes Θ, i.e. for all Θ passing through a dense open subset of G - the proof rests on a determinant identity (7.8.1) proven in Lemma 7.8.

(iv) The Yang-Mills quantum-field measure

$$d\mu_T^\Theta(\omega) = Z_T(\Theta)^{-1} e^{-S_{\mathrm{YM}}(\omega)/T} \delta_\Theta\left(h(C;\omega)\right)[\mathcal{D}\omega]$$

<u>converges</u>, as $T \downarrow 0$, <u>to the normalized symplectic volume measure</u> on $\mathcal{M}(\Theta)$. A precise statement and the notation will be explained later; the determinant identity (7.8.1) is again the key

(v) the symplectic volume of $\mathcal{M}(\Theta)$, in the case $G = SU(2)$, is computed in Theorem 9.1:

$$\mathrm{vol}_{\overline{\Omega}_\Theta}(\mathcal{M}(\Theta)) = \begin{cases} 2\pi(\pi-\theta)\left[\frac{\mathrm{vol}(SU(2))}{2\pi^2}\right]^{\frac{2}{3}} & \text{if } \mathrm{g}=1 \\ 4\pi\sin\theta\left[\frac{\mathrm{vol}(SU(2))}{2\pi^2}\right]^{\frac{2}{3}} \mathrm{vol}\,(SU(2))^{2\mathrm{g}-2}\sum_{n=1}^{\infty}\frac{\chi_n(c)}{n^{2\mathrm{g}-1}} & \text{if } \mathrm{g}\geq 2 \end{cases}$$

wherein c is any element of Θ, $\chi_n(c) = \sin n\theta/\sin\theta$, with θ specified by $\mathrm{Tr}\, c = 2\cos\theta$. (The formula for $\mathrm{g} \geq 2$ is also valid for $\mathrm{g} = 1$)

<u>*Related recent works*</u>. Recent interest in 2–dimensional quantum gauge theory, attested to by, for instance, the works [AIK, Be, Di, Fi, Fo, Je, KS1-3, RR, Se1-5, Wi1-3], stems in part from questions associated to a 'classical limit' of the quantum theory; in particular, in determining the relationship of the classical limit of the quantum field measure, over oriented surfaces, to a symplectic volume measure on the moduli space of flat connections. The results of Sects. 6–11 of the present paper address the natural extension of this question to the case where we consider connections over surfaces with boundary with the holonomies around the boundary components known up to conjugation. The investigation of the limiting quantum Yang-Mills measure arose in the case of closed surfaces in [Se1] and in Witten's papers [Wi1,2]; a description of some

of the questions in this area is given in [Wi3]. The work [Fo] of Forman, and [Se4,5] are also devoted to the case of closed oriented surfaces. The most influential early works on the symplectic structure on the moduli space of flat connections on closed oriented surfaces are by Atiyah and Bott [AB] and Goldman [Go]. It seems likely that the symplectic structure that is the subject of the present work is the same as the one obtained through group-cohomological techniques by Biswas and Guruprasad [BG].

In [Wi1], Witten gives formulas for the symplectic volumes of moduli spaces of flat connections over a surface with punctures, with holonomies around these punctures lying in specified conjugacy classes. For a genus g Riemann surface with one puncture, with the holonomy lying in a conjugacy class Θ, the volume given by Witten ((3.18) or (4.116) in [Wi1]) is

$$2\frac{1}{2^{g-1}\pi^{2g-1}}\sum_{n=1}^{\infty}\frac{\sin n\theta}{n^{2g-1}}$$

where θ is such that $\operatorname{Tr} a = 2\cos\theta$ for $a \in \Theta$. To compare this with our volume formula given in (v) above (or Theorem 9.1) we need to note that: (i) the metric on $SU(2)$ used in [Wi1] is given by $\langle a, b\rangle = -\operatorname{Tr}(ab)$, and (ii) the symplectic form used in [Wi1] is $\frac{1}{4\pi^2}$ times ours (Eq. (2.29) in [Wi1]). This inner-product on $SU(2)$ corresponds to taking $SU(2)$ to be a 3-sphere of radius $2^{1/2}$; its volume then is $2\pi^2(2^{1/2})^3$. Moreover, since Witten's symplectic form is $(4\pi^2)^{-1}$ times ours, and since $\mathcal{M}(\Theta)$ has dimension $6g-4$, we must multiply our volume formula by $(4\pi^2)^{-(6g-4)/2}$. Putting all these pieces together in the volume formula given in (v) above, we get

$$\begin{aligned}\frac{1}{(4\pi^2)^{(6g-4)/2}}4\pi\sin\theta\cdot 2\cdot(2\pi^2 2^{3/2})^{2g-2}\sum_{n=1}^{\infty}\frac{\sin n\theta/\sin\theta}{n^{2g-1}}\\ = 2\frac{1}{2^{g-1}\pi^{2g-1}}\sum_{n=1}^{\infty}\frac{\sin n\theta}{n^{2g-1}}\end{aligned}$$

in pleasant agreement with the expression given by Witten.

2. Notation and Background for Construction of the Yang-Mills Measure

2.1 The surface Σ as a quotient of the disk D. We shall work with a compact 2–dimensional Riemannian manifold Σ. In Sects. 3 and 4, Σ is a torus with one hole, i.e. with one boundary component. It will be convenient to view this Σ as a quotient of the disk $D = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1\}$ in the following way. Consider the path $t \mapsto x_t \stackrel{\text{def}}{=} (\cos 2\pi t, \sin 2\pi t)$ tracing out ∂D. If $r < s$ then $x_r x_s$ will denote the path $t \mapsto x_t$ with $t \in [r, s]$. Divide ∂D into arcs $K_1, K_2, \ldots, K_7$, where K_j is given by x_t with $t \in [\frac{j-1}{2\pi}, \frac{j}{2\pi}]$, $j \in \{1, \ldots, 7\}$. Identify K_1 with $\overline{K}_3$ (the reverse of K_3), K_2 with $\overline{K}_4$, and K_5 with $\overline{K}_7$, linearly. This yields the quotient space Σ and the quotient map $q : D \to \Sigma$. We shall equip Σ with the orientation which makes q orientation preserving. The point $o = q(O)$, where $O = (0, 0)$ is the center of D, will serve as a basepoint on Σ. The loops

$$A = q(x_0 O)q(K_1)q(O x_0),$$

$$B = q(x_0O)q(K_2)q(Ox_0),$$
$$C = q(x_0O)q(K_7K_6K_5)q(Ox_0)$$

generate $\pi_1(\Sigma, o)$ subject to the relation

$$C\overline{B}\,\overline{A}BA = I \tag{2.1.1}$$

wherein I is the identity in $\pi_1(\Sigma, o)$. The presence of K_5 and K_7 is not essential; however, when Σ is a general compact surface with more than one boundary component (as will be the case in section 5) then arcs like K_5 and K_7 must be included, and this is why we choose to keep them in our framework.

2.2 The G-bundle $\pi : P \to \Sigma$, space of connections $\mathcal{A}$, holonomies $h(\kappa;\omega)$, and curvature Ω^ω. We shall work with a principal G–bundle $\pi : P \to \Sigma$, where G is a compact connected Lie group with Lie algebra $\underline{g}$ having an Ad-invariant inner-product $\langle\cdot,\cdot\rangle_{\underline{g}}$ on it. The set of all connections on P will be denoted $\mathcal{A}$. The metrics on Σ and on $\underline{g}$ induce a metric on $\mathcal{A}$ in a standard way. Fix once and for all a basepoint $u \in \pi^{-1}(o)$. If $\kappa : [r, s] \to \Sigma$ is a path on Σ with $\kappa(r) = o$, and if $\omega \in \mathcal{A}$, then we denote by $\tau_\omega(\kappa)u$ the parallel translate of u along κ with respect to ω. Thus if κ is a loop based at o then the *holonomy* of ω around κ, with u as initial point, is $h(\kappa;\omega) \in G$ given by $\tau_\omega(\kappa)u = uh(\kappa;\omega)$. The curvature of $\omega \in \mathcal{A}$ will be denoted Ω^ω; thus $\Omega^\omega = d\omega + \frac{1}{2}[\omega,\omega]$.

2.3 The gauge transformation groups $\mathcal{G}$,$\mathcal{G}_0$, acting on $\mathcal{A}$. We shall use the set $\mathcal{G}$ of automorphisms of P, i.e. diffeomorphisms $\phi : P \to P$ for which $\pi \circ \phi = \pi$ and $\phi(pg) = \phi(p)g$ for all $p \in P$, $g \in G$. This is a group under composition, and the subgroup $\mathcal{G}_o \stackrel{\text{def}}{=} \{\phi \in \mathcal{G} : \phi(u) = u\}$ will be of use. These groups act on $\mathcal{A}$ by $(\phi,\omega) \mapsto \phi^*\omega$.

2.4 The Yang-Mills action. If $\omega \in \mathcal{A}$ then the Yang-Mills action $S_{\mathrm{YM}}(\omega)$ is given by

$$S_{\mathrm{YM}}(\omega) = \frac{1}{2}\int_\Sigma \|\Omega^\omega\|^2_{\underline{g}}\, d\sigma, \tag{2.4.1}$$

where $d\sigma$ is the Riemannian surface area measure on Σ, and $\|\Omega^\omega\|^2_{\underline{g}}$ is the function on Σ given by

$$\|\Omega^\omega\|^2_{\underline{g}}(x) = \|\Omega^\omega(e_1, e_2)\|^2_{\underline{g}},$$

where x runs over Σ, and $e_1, e_2 \in T_pP$, for any $p \in \pi^{-1}(x)$, are such that $(\pi_* e_1, \pi_* e_2)$ is an orthonormal basis of $T_x\Sigma$. Since Ω^ω vanishes when to applied vertical vectors and since it is a 2–form, it follows from the Ad-invariance of $\langle\cdot,\cdot\rangle_{\underline{g}}$ that $\|\Omega^\omega\|^2_{\underline{g}}$ is a well-defined function Σ. Furthermore, S_{YM} is invariant under the $\mathcal{G}$-action and therefore defines a function on $\mathcal{A}/\mathcal{G}$ and on $\mathcal{A}/\mathcal{G}_o$.

2.5 The curvature function F^ω, and the parallel-transport equation. Let $\omega \in \mathcal{A}$. We shall use the map $s_\omega : D \to P$ given by $s_\omega(x) = \tau_\omega\Big(q(Ox)\Big)u$, where Ox is the radial path from O to x. A convenient way to express the curvature is by means of the map

$$F^\omega : D \to \underline{g} : x \mapsto F^\omega(x) \stackrel{\text{def}}{=} \Omega^\omega(e_1, e_2), \tag{2.5.1}$$

where on the right $e_1, e_2 \in T_pP$, with $p = s_\omega(x)$, are such that $(\pi_1 e_1, \pi_* e_2)$ is a positively oriented orthonormal basis of $T_{q(x)}\Sigma$. (If Σ were an unorientable surface the orientation on $T_{q(x)}\Sigma$ here would be the one which would make q orientation-preserving in a neighborhood of x.) Then

$$S_{\mathrm{YM}}(\omega) = \frac{1}{2}\int_D \|F^\omega\|^2_{\underline{g}}\, d\sigma \tag{2.5.2}$$

with $d\sigma$ here being the surface area measure for Σ pulled up to D by q.

We shall almost always work with *admissible curves* on Σ; by an *admissible curve* we mean a curve of the form $q \circ \kappa$, where $\kappa : [0,1] \to D : t \mapsto \kappa_t$ is a path which can be expressed (i.e. reparametrized) in polar coordinates '$r = r(\theta)$' (thus κ cuts every radius, excluding O, at most once). If κ is such a path and if $q \circ \kappa$ is piecewise smooth, then for each $t \in [0,1]$ we have a loop $q(\kappa_t O) \cdot q \circ \kappa|[0,t] \cdot q(O\kappa_0)$; these loops will be very useful. The holonomy

$$h_t(\omega) \stackrel{\text{def}}{=} h\left(q(\kappa_t O) \cdot q \circ \kappa|[0,t] \cdot q(O\kappa_0); \omega\right) \tag{2.5.3}$$

satisfies the differential equation of parallel-transport:

$$dh_t(\omega)h_t(\omega)^{-1} = -d\left(\int_{D^{\kappa}{}_t} F^\omega\, d\sigma\right), \tag{2.5.4}$$

where $D^{\kappa}{}_t$ is the subset of D whose positive boundary is $\kappa_t O \cdot \kappa|[0,t] \cdot O\kappa_0$, i.e. $D^\kappa_t = \{rx_s : r \in [0, \kappa(s)], s \in [0,t]\}$.

2.6 The map $\omega \mapsto \left(F^\omega, h(A;\omega), h(B;\omega), h(C;\omega)\right)$. The map

$$\omega \mapsto \left(F^\omega, h(A;\omega), h(B;\omega), h(C;\omega)\right) \tag{2.6.1}$$

induces a one-to-one map from the quotient space $\mathcal{A}/\mathcal{G}_o$. However, the image is constrained by the condition (cf. (2.1.1))

$$h(C;\omega)h(B;\omega)^{-1}h(A;\omega)^{-1}h(B;\omega)h(A;\omega) = h(\omega(\overline{L}_0 \cdot \partial D \cdot L_0);\omega), \tag{2.6.2}$$

where L_0 is the radial path from O to $x_0 = (1,0)$, and $h(\omega(\overline{L}_0 \cdot \partial D \cdot L_0);\omega)$ is computable, by means of (2.5.4), in terms of only F^ω.

2.7 Yang-Mills measures μ_T, μ_T^c, μ_T^Θ for the spaces $\mathcal{A}$, $\mathcal{A}_c$, $\mathcal{A}_\Theta$. Let $T > 0$. The Yang-Mills measure for $\mathcal{A}$ is, informally, a probability measure μ_T on $\mathcal{A}/\mathcal{G}$ given by the heuristic formula

$$d\mu_T([\omega]) = \frac{1}{Z_T}e^{-S_{\mathrm{YM}}(\omega)/T}[\mathcal{D}\omega], \tag{2.7.1}$$

where $[\mathcal{D}\omega]$ is the formal Riemannian volume measure on $\mathcal{A}$ pushed down to $\mathcal{A}/\mathcal{G}$, and Z_T is a 'normalizing constant'. A rigorous construction of μ_T is given in [Se2].

Let $c \in G$, and consider

$$\mathcal{A}_c = \{\omega \in \mathcal{A} : h(C;\omega) = c\}, \tag{2.7.2}$$

If Θ is a conjugacy class in G, we consider also

$$\mathcal{A}_\Theta = \{\omega \in \mathcal{A} : h(C;\omega) \in \Theta\}. \tag{2.7.3}$$

Our goal is to construct a probability measure μ_c on (an appropriate completion of) $\mathcal{A}_c/\mathcal{G}_o$, and a probability measure μ_Θ on $\mathcal{A}_\Theta/\mathcal{G}_o$ which are given heuristically by

$$d\mu_T^c(\omega) = \frac{1}{Z_T(c)}\delta\left(h(C;\omega)c^{-1}\right)e^{-S_{\mathrm{YM}}(\omega)/T}[\mathcal{D}\omega] \text{ on } \mathcal{A}_c/\mathcal{G}_o, \tag{2.7.4}$$

$$d\mu_T^\Theta(\omega) = \frac{1}{Z_T(\Theta)}\delta_\Theta\left(h(C;\omega)\right)e^{-S_{\mathrm{YM}}(\omega)}[\mathcal{D}\omega] \text{ on } \mathcal{A}/\mathcal{G}, \tag{2.7.5}$$

wherein $Z_T(c)$ and $Z_T(\Theta)$ are "normalizing constants" and δ_Θ is the δ–function specified by "$\int_G f(x)\delta_\Theta(x)\,dx$" $= \int_\Theta f(x)\,d_\Theta(x) \stackrel{\text{def}}{=} \int_G f(k\theta k^{-1})\,dk$ for any $\theta \in \Theta$ (and dk is the unit-mass Haar measure on G).

Actually we will realize these measures on certain larger spaces Ω_c and Ω_Θ.

3. Construction of the Measures μ_T^c and μ_T^Θ

3.1 The Strategy. Since $S_{\mathrm{YM}}(\omega) = \frac{1}{2}\|F^\omega\|^2_{L^2(D;\underline{g})}$, the expression (2.7.5) for μ_T^Θ suggests that it is reasonable to construct μ_T^Θ as a Gaussian measure on $L^2(D;\underline{g})$ times Haar measure on G^3 (corresponding to the holonomies $h(A;\omega)$, $h(B;\omega)$ and $h(C;\omega)$) <u>conditioned</u> to satisfy the constraint (2.6.2) as well as the constraint $h(C;\omega) \in \Theta$ (recall that Θ is a conjugacy class in G). Similarly, for μ_T^c we must use the constraint $h(C;\omega) = c$ in addition to (2.6.2).

3.2 Stochastic parallel transport and holonomy over the disk D. As is well known, the standard Gaussian measure 'on' $L^2(D;\underline{g})$ actually lives on a Hilbert-Schmidt closure $\overline{L^2(D;\underline{g})}$. Henceforth we shall write Ω_{disk} for the space $\overline{L^2(D;\underline{g})}$, and $\mu_{\mathrm{T,disk}}$ for the Gaussian measure, with covariance scaled by $T > 0$, on it. The probability space for quantum gauge theory over the disk D is:

$$(\Omega_{\mathrm{disk}}, \mu_{\mathrm{T,disk}}) = \left(\overline{L^2(D;\underline{g})}, \mu_{\mathrm{T,disk}}\right)$$

For $\omega \in \Omega_{\mathrm{disk}}$, F^ω now corresponds to a $\underline{g}$–valued white-noise; i.e. to each Borel set $E \subset \mathbf{R}^2$ there is a Gaussian $\underline{g}$–valued random variable $\omega \mapsto \int_E F^\omega\,d\sigma \stackrel{\text{def}}{=} (F^\omega, 1_E)$; a more detailed account is given in Sect. A3 of the Appendix. In order to impose the constraint (2.6.2) the meaning of the holonomy $h(\overline{L}_0 \cdot \partial D \cdot L_0;\omega)$ needs to be understood for $\omega \in \Omega_{\mathrm{disk}}$. More generally we would like to (and will need) give appropriate meaning to the differential equation for parallel-transport (2.5.4).

Thus let $\kappa : [0,1] \to D$ be an admissible path, and define parallel-transport along κ by reinterpreting (2.5.4) as a *Stratonovich* stochastic differential equation (this idea is due to L. Gross). Thus in place of (2.5.4) we consider the Stratonovich stochastic differential equation

$$dh_t(\omega) \circ h_t(\omega)^{-1} = -d\left(\int_{D_t^\kappa} F^\omega\,d\sigma\right), \tag{3.2.1}$$

with initial condition $h_0(\omega) = e$, where D_t^κ is, as before, the region $\{rx_s : 0 \le r \le \kappa(s), 0 \le s \le t\}$.

The solution $h_t(\omega)$ of (3.2.1) can be obtained as a (probabilistic) limit of products of the form $\exp\left(\int_{D^\kappa_{t_j}\setminus D^\kappa_{t_{j-1}}} F^\omega\, d\sigma\right)$, where $t_0 = 0 < t_1 < \cdots < t_N = t$ and $\max |t_j - t_{j-1}| \to 0$.

Now $t \mapsto \int_{D_t^\kappa} F^\omega\, d\sigma$ is, in law, a $\underline{g}$—valued Brownian motion with time clocked by the quadratic variation which, given that F^ω is Gaussian as described before, is simply $T|D_t^\kappa|$, where $|D_t^\kappa|$ is the area of D_t^κ. Thus the solution of (3.2.1) is, in law, simply a Brownian motion on G with time clocked by $T|D_t^\kappa|$ instead of by t.

In particular, the density of $h_t(\cdot)$, with respect to unit-mass Haar measure on G, is $Q_{T|D_t^\kappa|}(\cdot)$, where $Q_t(x)$ is the heat kernel on G normalized to $\int_G Q_t(x)\,dx = 1$ (here dx is unit-mass Haar measure on G).

Furthermore, it is proven in [Se2,3] that if κ_1 and κ_2 are admissible loops in D whose interiors do not overlap then $h(\kappa_1;\cdot)$ and $h(\kappa_2;\cdot)$ are independent G-valued random variables on the Gaussian probability space $(\Omega_{\text{disk}}, \mu_{\text{T,disk}})$.

Thus, if $\kappa_1, \ldots, \kappa_n$ are non-overlapping admissible loops, based at O, in D then the joint distribution measure, under $\mu_{\text{T,disk}}$, of $\left(h(\kappa_1;\omega), \ldots, h(\kappa_n;\omega)\right)$, as a random variable in ω running over Ω_{disk}, is

$$Q_{TA_1}(y_1)\cdots Q_{TA_n}(y_n)\,dy_1\cdots dy_n\,, \tag{3.2.2}$$

where A_i is the area enclosed by κ_i and dy_i is unit-mass Haar measure on G.

3.3 Construction of the measures μ_c^T and μ_T^Θ. The construction of the conditional probability, satisfying the constraint (2.6.2), requires the technical artifice of dividing D into a 'lower-half' D_L and an 'upper-half' D_U and working with $h(\overline{L}_0 \cdot \partial D \cdot L_0;\omega) = h(\partial D_L;\omega)h(\partial D_U;\omega)$. The full technical details of this construction in a general setting are presented in [Se2,3] and so we shall give here only a conceptual account. In particular, we shall not make explicit the technical role played by D_L and D_U. In the Appendix we shall quote the relevant tools from [Se2,3] and explain how they apply in the present context.

As explained in the Appendix (Sect. A3), for any $c \in G$ there is a probability measure μ_T^c on the space

$$\Omega_c \stackrel{\text{def}}{=} \Omega_{\text{disk}} \times G^2 \tag{3.3.1}$$

which satisifies (cf. the constraint (2.6.2))

$$h(\overline{L}_0 \cdot \partial D \cdot L_0;\omega) = cb^{-1}a^{-1}ba \quad \text{for } \mu_T^c\text{-a.e. } \omega = (\omega_1, a, b) \in \Omega_c \tag{3.3.2}$$

and is, in a natural and precise sense, the conditioning of the probability measure

$$\mu_{\text{T,disk}} \times (\text{Haar on } G^2)$$

to satisfy the constraint (3.3.2).

Analogously, for any conjugacy class Θ in G, there is a probability measure μ_T^Θ on

$$\Omega_\Theta \stackrel{\text{def}}{=} \Omega_{\text{disk}} \times G^2 \times \Theta \tag{3.3.3}$$

such that

$$h(\overline{L}_0 \cdot \partial D \cdot L_0;\omega) = cb^{-1}a^{-1}ba \text{ for } \mu_T^\Theta\text{-a.e. } \omega = (\omega_1, a, b, c) \in \Omega_\Theta \tag{3.3.4}$$

and μ_T^Θ is the conditioning of the probability measure

$$\mu_{\mathrm{T,disk}} \times (\text{Haar on } G^2) \times (G\text{–invariant measure on } \Theta)$$

to satisfy the constraint (3.3.4).

3.4 Expectation values. Consider a measurable function $\phi = (\phi_1, ..., \phi_n) : \Omega_{\mathrm{disk}} \to G^n$ which has a bounded density ρ_ϕ on G^n, and consider a bounded measurable function f on $G^n \times G^2$. Suppose $\phi_k ... \phi_1 = h(\overline{L}_0 \cdot \partial D \cdot L_0;\ \cdot)$ for some $k \in \{1, ..., n\}$. Then under simple conditions (detailed in Proposition A6 and the discussion following it) on ϕ and ψ:

$$\int_{\Omega_c} f(\phi(\omega_1), a, b)\, d\mu_T^c(\omega_1, a, b) = \frac{1}{N_T(c)} \int_{G^2} da\, db \int_{G^n} dx\, f(x, a, b)\rho_\phi(x)\delta\Big(x_k \cdots x_1 (cb^{-1}a^{-1}ba)^{-1}\Big), \quad (3.4.1)$$

where da and db are the unit-mass Haar measure on G, $dx = dx_1 \cdots dx_n$, and

$$N_T(c) = \int_{G^2} Q_{T|\Sigma|}(cb^{-1}a^{-1}ba)\, da\, db \quad (3.4.2)$$

with Q_t being the heat kernel on G, as in Sect. 3.2, and $|\Sigma|$ the area of Σ. The significance of the 'delta function' $\delta(x_k \cdots x_1 \cdots)$ in (3.4.1) is simply that one of the variables x_j should be replaced by the value which makes the product $x_k \cdots x_1 \cdots$ equal to the identity (and the integration dx_j dropped).

The corresponding result for Ω_Θ is:

$$\int_{\Omega_\Theta} f(\phi(\omega_1), a, b)\, d\mu_T^\Theta(\omega_1, a, b, c) = \frac{1}{N_T(\Theta)} \int_\Theta d_\Theta c \int_{G^2} da\, db \int_{G^n} dx\, f(x, a, b)\rho_\phi(x)\delta\Big(x_k \cdots x_1 (cb^{-1}a^{-1}ba)^{-1}\Big), \quad (3.4.3)$$

where $d_\Theta c$ is the unit-mass , G–invariant measure on the conjugacy class Θ specified by the integration formula

$$\textstyle\int_\Theta F(c)\, d_\Theta c = \int_G F(kck^{-1})\, dk \text{ for any } c \in \Theta \quad (3.4.4)$$

and the normalizer $N_T(\Theta)$ is given by:

$$N_T(\Theta) = \int_{G^2} Q_{T|\Sigma|}(cb^{-1}a^{-1}ba)\, da\, db\, d_\Theta c \quad (3.4.5)$$

Thus $N_T(\Theta) = N_T(c)$ if $c \in \Theta$.

These results are taken from Proposition 4.5 of [Se2]; for ease of reference, we quote the exact result in the Appendix (Sect. A6).

The expectation value formulas (3.4.1) and (3.4.4) follow from Proposition A6 (quoted from [Se2,3]) in the manner explained in the discussion following A6 in the Appendix.

4. Loop Expectation Values

4.1 Triangulation of Σ. We shall always work with a certain type of triangulation ('admissible triangulation') $\mathcal{S}$ of Σ obtained by means of the quotient map $q : D \to \Sigma$ from a triangulation $\mathcal{T}$ of D. The triangulation $\mathcal{T}$ of D will consist of radii and cross-radial segments (i.e. those which can be parametrized in polar coordinates in the form $r = r(\theta)$), and will include the origin as a vertex. We will also assume that the arcs on ∂D corresponding to the loops A, B and C are made up of 1−simplices of the triangulation (this can always be ensured by subdividing the original triangulation). These technical restrictions are to ensure that the parallel-transport equation (3.2.1) is applicable and so that then the relevant holonomies are meaningful.

4.2 G−fields and lassos. An assignment $e \mapsto y_e \in G$, with e running over the oriented 1−simplices of a triangulation, satisfying $y_{\bar{e}} = y_e^{-1}$ will be called a *G−field over the triangulation.* If κ is a path consisting of oriented 1−simplices of the triangulation, $\kappa = b_j \cdots b_1$, then we define

$$y(\kappa) \stackrel{\text{def}}{=} y_{b_j} \cdots y_{b_1}.$$

Let $\mathcal{T}$ be a triangulation of D as described in Sect. 4.1, and Δ a positively oriented 2−simplex (triangle) in $\mathcal{T}$. We can join an appropriate vertex of Δ to the origin O by a radial path consisting of 1−simplexes of $\mathcal{T}$ and thus form a loop l_Δ based at O. In this way we form loops $l_{\Delta_1}, \dots, l_{\Delta_m}$, one for each positively oriented 2−simplex Δ_j in $\mathcal{T}$. The loops Δ_i will be called the *lassos* of the triangulation. This can be done, with the Δ_i ordered appropriately, in such a way that the following hold :

(*) For any G−field y over $\mathcal{T}$:

$$y(l_{\Delta_m}) \cdots y(l_{\Delta_1}) = y(x_0 O) \cdot y(\partial D) \cdot y(O x_0), \tag{4.2.1}$$

where ∂D is taken to start from $x_0 = (1, 0) \in \partial D$. A complete description of the construction of the loops l_{Δ_i} is given in Theorem 3.1, and the discussion preceding it, of [Se2].

(**) To each loop κ in $\mathcal{T}$, based at o, we can associate a sequence of the lassos $l_{\Delta_{i_1}}, l_{\Delta_{i_2}} \dots$ such that $y(\kappa) = y(l_{\Delta_{i_1}})^{\pm 1} y(l_{\Delta_{i_2}})^{\pm 1} \cdots$ holds for any G−field y over $\mathcal{T}$.

These are simple graphical facts and are actually independent of considerations of G−fields (Lemma A3 in [Se2]).

4.3 Stochastic holonomy over Σ, for μ_T^c and μ_T^Θ. We shall work with loops on Σ based at O, which consist of oriented 1−simplices of some triangulation $\mathcal{S}$ of Σ of the type described in Sect. 4.1. If κ is such a loop then there exists a sequence of loops $\kappa_1, \dots, \kappa_r$ such that:

(a) each κ_j is either the projection by q of a loop $\tilde{\kappa}_j$ in D or is one of the basic loops A, B, C and their reverses,

(b) for any G−field y over $\mathcal{S}$, $y(\kappa_r) \cdots y(\kappa_1) = y(\kappa)$.

The existence of such a decomposition is seen by breaking κ into suitable segments and lifting these up to D (a detailed account is given in Sect. 5 of [Se2]). Recall that for the loops $\tilde{\kappa}_j$ on D the stochastic holonomy $h(\tilde{\kappa}_j;\omega) \in G$ is meaningful as a random variable in $\omega \in \Omega_{\text{disk}}$.

For $\omega = (\omega_1, a, b) \in \Omega_c$, or for $\omega = (\omega_1, a, b, c) \in \Omega_\Theta$, we define

$$h(\kappa;\omega) \stackrel{\text{def}}{=} h(\kappa_r;\omega)\cdots h(\kappa_1;\omega), \tag{4.3.1}$$

where

$$h(\kappa_j;\omega) = \begin{cases} h(\tilde{\kappa}_j;\omega_1) & \text{if } \tilde{\kappa}_j \text{ is the loop in } D \text{ projecting to } \kappa_j \\ a^{\pm 1} & \text{if } \kappa_j \text{ is } A \text{ or } \overline{A} \\ b^{\pm 1} & \text{if } \kappa_j \text{ is } B \text{ or } \overline{B} \\ c^{\pm 1} & \text{if } \kappa_j \text{ is } C \text{ or } \overline{C} \end{cases} . \tag{4.3.2}$$

Then $h(\kappa;\omega)$ is a well-defined G–valued random variable on Ω_c and Ω_Θ (that $h(\kappa;\omega)$ is independent of the choice of the κ_j is proven in Lemma A2 of [Se2]).

4.4.

Theorem 1. *Let $\kappa_1, \ldots, \kappa_n$ be loops on Σ, all based at o, composed of oriented 1–simplices of a triangulation S of Σ of the type described in Sect. 4.1). Let f be a bounded measurable function on G^n. Consider any $c \in G$. Then*

$$\begin{aligned} & f\left(h(\kappa_1;\omega), \ldots, h(\kappa_n;\omega)\right) d\mu_T^c(\omega) \\ &= \frac{1}{N_T(c)} \int f\left(x(\kappa_1), \ldots, x(\kappa_n)\right) \cdot \\ &\cdot \delta\left(x(C)c^{-1}\right) \prod_{j=1}^{m} Q_{T|\Delta_j|}\left(x(\partial\Delta_j)\right) dx_{e_1} \cdots dx_{e_M}, \end{aligned} \tag{4.4.1}$$

where $\Delta_1, \ldots, \Delta_m$ are the positively oriented 2–simplexes of S, $|\Delta_j|$ is the area of Δ_j, $e_1, \ldots, e_M$ and their reverses are the distinct oriented 1–simplices of S, $Q_t(\cdot)$ is the heat kernel on G normalized to $\int_G Q_t(x)\,dx = 1$, dx being unit-mass Haar measure on G, and

$$N_T(c) = \int_{G^2} Q_{T|\Sigma|}(cb^{-1}a^{-1}ba)\, da\, db. \tag{4.4.2}$$

The meaning of the δ–function in (4.4.1) *is that for some arbitrary bond e_j lying on $\partial\Sigma$ and appearing in the loop C, the variable x_{e_j} should be replaced by the value which makes $x(C) = c$ and dx_{e_j} should be dropped from the integration.*

Proof. In view of the observation (**) concerning the decomposition of a loop in D in terms of the lassos l_{Δ_i}, and in view of the definition of $h(\kappa;\omega)$ in (4.3.1), we see that each $h(\kappa_i;\omega)$ is a product of certain of the $h(l_{\Delta_i};\omega)^{\pm 1}$'s, $h(A;\omega)^{\pm 1}$, $h(B;\omega)^{\pm 1}$, and $h(C;\omega)^{\pm 1}$ (the latter being $c^{\pm 1}$).

Thus to f we may associate a bounded measurable function F such that

$$f\left(h(\kappa_1;\omega), \ldots, h(\kappa_n;\omega)\right) = F\left(h(l_{\Delta_1};\omega), \ldots, h(l_{\Delta_m};\omega), h(A;\omega), h(B;\omega), h(C;\omega)\right) \tag{4.4.3}$$

and, more generally,

$$f\left(x(\kappa_1), \ldots, x(\kappa_n)\right) = F\left(x(l_{\Delta_1}), \ldots, x(l_{\Delta_m}), a, b, c\right) \tag{4.4.4}$$

for any G–field x over the triangulation $\mathcal{S}$ (of course, the c–term, being constant, could be dropped from F but that would make the form of F dependent on c).

Recalling the μ_c–expectation-value formula (3.4.1) and the μ–distribution of the random variables $h(l_{\Delta_i};\cdot)$ given in (3.2.2), we have:

$$\begin{aligned}
&\int f\left(h(\kappa_1;\omega),\ldots,h(\kappa_n;\omega)\right)\,d\mu_T^c(\omega)\\
&=\frac{1}{N_T(c)}\int_{G^n\times G^2} F\left(y_{\Delta_1},\ldots,y_{\Delta_m},a,b,c\right)\cdot\\
&\cdot\delta\left(y_{\Delta_m}\cdots y_{\Delta_1}(cb^{-1}a^{-1}ba)^{-1}\right)\,da\,db\prod_{j=1}^{m}Q_{T|\Delta_j|}(y_{\Delta_j})dy_{\Delta_j},
\end{aligned}\tag{4.4.5}$$

where $N_T(c)$ is as in (4.4.2) (which is the same as stated earlier in (3.4.2)).

A combinatorial argument (Lemma A1 in [Se2]) shows that if we take the y_{Δ_i}'s and a, b, c such that $y_{\Delta_m}\cdots y_{\Delta_1} = cb^{-1}a^{-1}ba$ (as is required by the $\delta(\cdot)$ term in (4.4.5)) then it is possible to associate to this data a G–field x over $\mathcal{S}$ in such a way that

$$x(l_{\Delta_i}) = y_{\Delta_i} \text{ for } i \in \{1,\ldots,m\}, \tag{4.4.6a}$$

$$x(A) = a,\ x(B) = b,\ x(C) = c. \tag{4.4.6b}$$

In (4.4.6*a*) we have tacitly raised x to a G–field, also denoted x, on the triangulation of D which projects by $q : D \to \Sigma$ to the triangulation $\mathcal{S}$ of Σ.

The goal now is to change variables $(\{y_{\Delta_i}\}, a, b) \mapsto \{x_{e_i}\}$. In order to do this it is necessary to understand how $\prod_{i=1}^{m} dy_{\Delta_i}\,da\,db$ is transformed. It is shown in the proof of Lemma A1 in [Se2] that the G–field x can be chosen in the following way:

(i) for a certain collection of bonds e, the variables x_e are chosen arbitrarily; for the remaining bonds :

(ii) for one bond e_A^* lying on $q(\partial D)$ and appearing in the loop A (i.e. e_A^* lies on the part of A on $q(\partial D)$), $x(e_A^*)$ is the product of a with some of the 'previously' assigned values of x;

(iii) for one bond e_B^* lying on $q(\partial D)$ and appearing in the loop B, $x(e_B^*)$ is product of b with some of the 'previously' assigned values of x;

(iv) for one bond e_C^* lying on $\partial\Sigma$ and appearing in the loop C, $x(e_C^*)$ is product of c with some of the 'previously' assigned values of x;

(v) there is one bond b_i corresponding to each Δ_i with x_{b_i} chosen to be y_{Δ_i} times some of the 'previously' assigned values of x.

(A more precise formulation, along with a specification of what 'previously assigned' means, is given in the proof of Lemma A1 of [Se2].)

From (i)–(v), and the translation-invariance of Haar measure, it follows that in the integral on the right side of (4.4.5), we can introduce the variables $\{x_{e_j}\}$ instead of $(\{y_{\Delta_i}\}, a, b)$, and then the right side of (4.4.5) equals:

$$\begin{aligned}
&\frac{1}{N_T(c)}\int F\left(x(\Delta_1),\ldots,x(\Delta_m),x(A),x(B),c\right)\cdot\\
&\cdot\delta\left(x(C)c^{-1}\right)\prod_{j=1}^{m}Q_{T|\Delta_j|}\left(x(\partial\Delta_j)\right)\,dx_{e_1}\cdots dx_{e_M}.
\end{aligned}\tag{4.4.7}$$

This calls for some explanation: recall from (i) that certain x_{e_i} can be chosen arbitrarily, these are integrated over G in (4.4.7) without changing the value of the integral since $\int_G dx = 1$; the term $\delta\left(y_{\Delta_m} \cdots y_{\Delta_1}(cb^{-1}a^{-1}ba)^{-1}\right)$ disappeared since (4.4.6a, b) and (4.2.1) imply that $y_{\Delta_n} \cdots y_{\Delta_1}(cb^{-1}a^{-1}ba)^{-1} = e$ is satisfied automatically by our choice of the x_{e_i}. Finally the new $\delta(\cdot)$–term $\delta\left(x(C)c^{-1}\right)$ appears because, by condition (iii) above, one of the bond variables x_e, with $e = e_C^*$ on $\partial\Sigma$ and appearing in C, must be chosen in such a way that $x(C) = c$; this says that this x_e is not really a 'variable' but is determined by the other x_{e_i} (and c). That the choice of e_C^* does not affect the value of the integral (4.4.7) follows by a change-of-variables argument (if e'_C is another bond on $\partial\Sigma$ appearing in C then, given all the other variables x_e, a new variable $x(e'_C)$ may be introduced in place of $x(e_C^*)$ by requiring that $x(e'_C)$ be such that $x(C) = c$, a relationship that expresses $x(e'_C)$ as a product of $x(e_C^*)$ with certain of the other bond-variables x_e).

Now (4.4.4) and (4.4.6b) imply that (4.4.7) is equal to

$$\frac{1}{N_T(c)} \int f\left(x(\kappa_1), \ldots, x(\kappa_n)\right) \delta\left(x(C)c^{-1}\right) \prod_{i=j}^{m} Q_{T|\Delta_j|}\left(x(\partial\Delta_j)\right) dx_{e_1} \cdots dx_{e_M}. \tag{4.4.8}$$

Thus, recalling that we started our algebraic manipulations with the loop expectation-value formula (4.4.5), we see that $\int f\left(h(\kappa_1;\omega), \ldots, h(\kappa_n;\omega)\right) d\mu_T^c(\omega)$ is equal to (4.4.8), thereby proving (4.4.1). □

4.5 Invariance of loop expectation values for μ_T^c. Unlike in the case of the "free-boundary" theory developed in [Se2], we cannot expect the loop expectation values

$$\int f\left(h(\kappa_1;\omega), \ldots, h(\kappa_n;\omega)\right) d\mu_T^c(\omega)$$

to be invariant under area-preserving transformations of Σ. This is because the loop C has been selected out as a special loop and so we must consider only area-preserving homeomorphisms of Σ which fix C.

Thus we wish to show that if ϕ is an area-preserving homeomorphism of Σ and $\phi \circ C = C$ then

$$\begin{aligned}
&\int f\left(x(\kappa_1), \ldots, x(\kappa_n)\right) \delta\left(x(C)c^{-1}\right) \cdot \\
&\quad \cdot \prod_{j=1}^{m} Q_{T|\Delta_j|}\left(x(\partial\Delta_j)\right) dx_{e_1} \cdots dx_{e_M} \\
&= \int f\left(x(\phi\circ\kappa_1), \ldots, x(\phi\circ\kappa_n)\right) \delta\left(x(C)c^{-1}\right) \cdot \\
&\quad \cdot \prod_{j=1}^{m'} Q_{T|\Delta'_j|}\left(x(\partial\Delta'_j)\right) dx_{e'_1} \cdots dx_{e'_{M'}},
\end{aligned} \tag{4.5.1}$$

where on the left we are using a triangulation $\mathcal{S}$ while on the right we are using a triangulation $\mathcal{S}'$. The loops κ_i and C are composed of simplices in $\mathcal{S}$, and the loops $\phi \circ \kappa_i$ and C are composed of simplices in $\mathcal{S}'$.

First we observe that either side of (4.5.1) is invariant under subdivisions of the triangulation. This is a consequence of the convolution property of the heat kernel $Q_t(\cdot)$:

$$\int_G Q_t(xy^{-1})Q_s(yz)\,dy = Q_{t+s}(xz),$$

which we can use to 'collapse' adjacent 2–simplices Δ_1^* and Δ_2^* of a subdivision of $\mathcal{S}$:

$$\int Q_{T|\Delta_1^*|}\Big(x(\partial\Delta_1^*)\Big)Q_{T|\Delta_2^*|}\Big(x(\partial\Delta_2^*)\Big)\,dx_b = Q_{T|\Delta^*|}\Big(x(\partial\Delta^*)\Big) \tag{4.5.2}$$

wherein Δ^* is the region formed by collapsing Δ_1^* and Δ_2^* along their common edge b. Extra bond-variables which remain after collapsing such common edges can be integrated away (or combined into bond variables for the undivided triangulation) using $\int dx_b = 1$.

Since the only way area appears in the right of (4.5.2) is through $|\Delta^*| = |\Delta_1^*|+|\Delta_2^*|$, it follows that the left side of (4.5.1) depends on areas only through the areas of the connected components $\mathcal{R}_1,\ldots,\mathcal{R}_N$ of $\Sigma\setminus\cup_{i=1}^{n+1}\mathrm{Im}(\kappa_i)$ (here $\mathrm{Im}(\kappa_i)$ is the image of the path κ_i as a subset of Σ, and we have set $\kappa_{n+1}\stackrel{\mathrm{def}}{=}C$, for convenience). A more detailed argument is available in Fact 1 of the Appendix in [Se2].

By the '*Hauptvermutung* of topology' (Theorem 4.6 in [Br]), $\mathcal{S}$ and $\mathcal{S}'$ have subdivisions $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{S}}'$ and there is a simplicial isomorphism $\tilde\phi:\tilde{\mathcal{S}}\to\tilde{\mathcal{S}}'$ obtained by an isotopy of ϕ, preserving the sets $\mathrm{Im}(\kappa_i)$ and $\mathrm{Im}(C)$, such that $\tilde\phi\circ C = C$, and $\tilde\phi\circ\kappa_i$ and $\phi\circ\kappa_i$ are the same when taken as sequences of bonds in $\tilde{\mathcal{S}}'$. In view of the invariance of the integrals in consideration under subdivisions we may and will assume that $\tilde{\mathcal{S}}=\mathcal{S}$ and $\tilde{\mathcal{S}}'=\mathcal{S}'$. Thus $\tilde\phi$ is a simplicial isomorphism between $\mathcal{S}$ and $\mathcal{S}'$.

By simple relabelling of x_e as $x_{\tilde\phi(e)}$, the left side of (4.5.1) equals:

$$\int f\left(x(\tilde\phi\circ\kappa_1),\ldots,x(\tilde\phi\circ\kappa_n)\right)\delta\Big(x(\tilde\phi\circ C)c^{-1}\Big)\cdot \\ \cdot\prod_{j=1}^{m} Q_{T|\Delta_j|}\Big(x\big(\partial\tilde\phi(\Delta_j)\big)\Big)\,dx_{e_1'}\cdots dx_{e_M'}. \tag{4.5.3}$$

Since $\tilde\phi\circ\kappa_i$ and $\phi\circ\kappa_i$ are the same as sequences of bonds in $\mathcal{S}'$, and since $\tilde\phi\circ C = C$, relabelling the simplices makes (4.5.3) equal to:

$$\int f\left(x(\phi\circ\kappa_1),\ldots,x(\phi\circ\kappa_n)\right)\delta\Big(x(C)c^{-1}\Big) \\ \prod_{j=1}^{m} Q_{T|\tilde\phi^{-1}(\Delta_j')|}\Big(x(\partial\Delta_j')\Big)\,dx_{e_1'}\cdots dx_{e_{M'}'}, \tag{4.5.4}$$

(the number M' of edges in $\mathcal{S}'$ equals the number M because in the present situation $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic). Now, as observed earlier (after (4.5.2)), (4.5.4) depends on the areas $|\tilde\phi^{-1}(\Delta_j)|$ only through certain of their sums; specifically, through the areas of the regions $\tilde\phi^{-1}\phi(\mathcal{R}_i)$, where, as before, the $\mathcal{R}_1,\ldots,\mathcal{R}_N$ are the connected components of the complement of $\mathrm{Im}(k_1)\cup\cdots\cup\mathrm{Im}(k_n)\cup\mathrm{Im}(C)$ in Σ. Since $\tilde\phi$ is an isotopy of ϕ through maps taking the subset $\mathrm{Im}(k_1)\cup\cdots\cup\mathrm{Im}(k_n)\cup\mathrm{Im}(C)$ always into the same subset, it follows that $\tilde\phi^{-1}\phi(\mathcal{R}_i)=\mathcal{R}_i$. Therefore, (4.5.4) equals

$$\int f\left(x(\phi\circ\kappa_1),\ldots,x(\phi\circ\kappa_n)\right)\delta\left(x(C)c^{-1}\right)$$
$$\prod_{j=1}^{m'} Q_{T|\Delta'_j|}\left(x(\partial\Delta'_j)\right)\,dx_{e'_1}\cdots dx_{e'_{M'}} \tag{4.5.5}$$

and this being the right side of (4.5.1), we have proven (4.5.1).

4.5.1. Observations.

(1) The above arguments show that the restriction that ϕ is the identity on C may be removed if C were to be replaced by $\phi(C)$ on the right side of (4.5.1).
(2) Since the left side of (4.5.1) involves a triangulation S and the right side involves S', we may set ϕ to be the identity and conclude that either side is *independent of the specific triangulation* used.
(3) The observation (2) is not surprising in view of the fact that the integrals in (4.5.1) are, up to the constant $N_T(c)$, the loop expectation-value as given in Theorem 4.4. Nevertheless, the arguments above give an independent and direct proof (one which also does not depend on the triangulations being of the type described in Sect. 4.1).

We turn now to the corresponding results for μ_T^{Θ}. For this we use the notation $d_\Theta c$ to denote the G–invariant unit-mass measure on a conjugacy class Θ in G; thus

$$\int_\Theta F(c)\,d_\Theta c = \int_G F(kck^{-1})\,dk \text{ for any } c\in\Theta.$$

4.6.

Theorem 2. *Let $\kappa_1,\ldots,\kappa_n$ be loops on Σ, all based at o, composed of oriented 1–simplices of a good triangulation S of Σ (of the type described in Sect. 4.1). Let f be a bounded measurable function on G^n. Consider any conjugacy class Θ in G. Then*

$$\int f\left(h(\kappa_1;\omega),\ldots,h(\kappa_n;\omega)\right)\,d\mu_T^{\Theta}(\omega)$$
$$=\frac{1}{N_T(\Theta)}\int f\left(x(\kappa_1),\ldots,x(\kappa_n)\right)\delta\left(x(C)c^{-1}\right)$$
$$\prod_{j=1}^{m} Q_{T|\Delta_j|}\left(x(\partial\Delta_j)\right)\,dx_{e_1}\cdots dx_{e_M}d_\Theta c, \tag{4.6.1}$$

where $\Delta_1,\ldots,\Delta_m$ are the positively oriented 2–simplexes of S, $e_1,\ldots,e_M$ and their reverses are the distinct oriented 1–simplices of S, $Q_t(\cdot)$ is the heat kernel on G normalized to $\int_G Q_t(x)\,dx = 1$, dx being unit-mass Haar measure on G, and

$$N_T(\Theta)=\int_{G^2} Q_{T|\Sigma|}(cb^{-1}a^{-1}ba)\,da\,db\,d_\Theta c. \tag{4.6.2}$$

The meaning of the δ–function in (4.6.1) is that for some arbitrary bond e_j lying on $\partial\Sigma$ and appearing in the loop C, the variable x_{e_j} should be replaced by the value which makes $x(C)=c$ and dx_{e_j} should be dropped from the integration.

Proof. The proof is virtually identical to that of Theorem 4.4, except that we use the μ_T^{Θ}-expectation-value formula (3.4.3) as starting point instead of the μ_T^{c}−formula (3.4.1) used for proving Theorem 4.4. □

4.7.

Theorem 3. *The loop expectation values $\int f\,(h(\kappa_1;\omega),\ldots,h(\kappa_n;\omega))\,d\mu_T^{\Theta}(\omega)$ remain invariant if each κ_i is replaced by $\phi\circ\kappa_i$, where ϕ is any area-preserving homeomorphism of Σ which preserves the orientation of C.*

Proof. Recall, from Observation (1) in Sect. 4.5, that the general form of (4.5.3) is:

$$\begin{aligned}&\int f\,(x(\kappa_1),\ldots,x(\kappa_n))\,\delta\Big(x(C)c^{-1}\Big)\prod_{i=1}^{m}Q_{T|\Delta_j|}\Big(x(\partial\Delta_j)\Big)\,dx_{e_1}\cdots dx_{e_M}\\&=\int f\,(x(\phi\circ\kappa_1),\ldots,x(\phi\circ\kappa_n))\,\delta\Big(x(\phi\circ C)c^{-1}\Big)\cdot\\&\cdot\prod_{j=1}^{m'}Q_{T|\Delta'_j|}\Big(x(\partial\Delta'_j)\Big)\,dx_{e'_1}\cdots dx_{e'_{M'}},\end{aligned}\tag{4.7.1}$$

the notation being as for (4.5.1), and c any element of G. We will show that either integral depends not on the full curve C but only on the part of C which lies on $\partial\Sigma$.

In (4.7.1) we can take $c\in\Theta$ and integrate by the G−conjugation-invariant measure $d_\Theta c$; this yields a formula of the form:

$$\int[\cdots]\delta\Big(x(C)c^{-1}\Big)[\cdots]\,d_\Theta c=\int[\cdots]'\delta\Big(x(\phi\circ C)c^{-1}\Big)[\cdots]'\,d_\Theta c.\tag{4.7.2}$$

Now the loop C can be expressed as

$$C=\overline{L}\cdot C_\partial\cdot L,\tag{4.7.3}$$

where C_∂ lies entirely on $\partial\Sigma$, and L is a path from o to a point on $\partial\Sigma$ where C 'first hits' $\partial\Sigma$. Then

$$x(C)=x(L)^{-1}x(C_\partial)x(L).\tag{4.7.4}$$

We substitute this into (4.7.2) and observe that the $\delta(\cdots)$ term on the left can be reordered into the form

$$\delta\left(x(C_\partial)x(L)c^{-1}x(L)^{-1}\right).\tag{4.7.5a}$$

Now since $d_\Theta c$ is a G−conjugation-invariant measure on Θ it follows that in the integrations in (4.7.2), the δ−term (4.7.5*a*) can be replaced by

$$\delta\left(x(C_\partial)c^{-1}\right).\tag{4.7.5b}$$

Returning to (4.7.2), we then have

$$\int[\cdots]\delta\Big(x(C_\partial)c^{-1}\Big)[\cdots]\,d_\Theta c=\int[\cdots]'\delta\Big(x(\phi\circ C_\partial)c^{-1}\Big)[\cdots]'\,d_\Theta c.\tag{4.7.6}$$

Now C_∂ and $\phi\circ C_\partial$ can differ only in their starting points and orientation. We have assumed that $\phi\circ C$ and C have the same orientation; thus the loops C_∂ and $\phi\circ C_\partial$

can only differ in their starting points. But then $x(\phi \circ C_\partial)$, as a product of the bond-variables x_e, is a cyclic permutations of $x(C_\partial)$, i.e. a conjugate of $x(C_\partial)$. Since $d_{\Theta} c$ is G–conjugation-invariant it follows that in the right side of (4.7.6) $\phi \circ C_\partial$ can be replaced by C_∂. Tracing our arguments back to (4.7.1) we conclude that:

$$x(C)c^{-1}\Big) \prod_{j=1}^{m} Q_{T|\Delta_j|}\Big(x(\partial\Delta_j)\Big)\, dx_{e_1} \cdots dx_{e_M}$$

$$= \int f\left(x(\phi \circ \kappa_1), \ldots, x(\phi \circ \kappa_n)\right) \delta\Big(x(C)c^{-1}\Big)$$

$$\prod_{j=1}^{m'} Q_{T|\Delta'_j|}\Big(x(\partial\Delta'_j)\Big)\, dx_{e'_1} \cdots dx_{e'_{M'}}, \tag{4.7.7}$$

and this what we wished to prove. Note that, as in Sect. 4.5, taking ϕ to be the identity map in (4.7.7) shows that the loop-expectation-value formulas are actually independent of the particular triangulation used. □

5. Other Surfaces

In this section we will sketch the construction of the boundary-holonomy-restricted quantum gauge field measures for a general compact 2–dimensional Riemannian manifold with boundary, and write down the loop expectation-value formulas.

5.1 Hypotheses on Σ, and generators of $\pi_1(\Sigma, o)$. In this section, Σ is a compact 2–dimensional Riemannian manifold with boundary.

Instead of the loops A, B, C of the earlier sections, we now have the following generators of $\pi_1(\Sigma, o)$:

(Σ1) If Σ is orientable, has genus g ≥ 0, and has p boundary-components, then we have loops

$$A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_p$$

generating $\pi_1(\Sigma, o)$ subject to the relation that

$$C_p \cdots C_1 \overline{B}_g \overline{A}_g B_g A_g \cdots \overline{B}_1 \overline{A}_1 B_1 A_1 \text{ gives the identity in } \pi_1(\Sigma, o).$$

The loops C_i are of the form $\overline{L}_i C'_i L_i$, where C'_i traces a loop around boundary component #i and L_i is a simple path from o to a point of C'_i (the point where C_i 'first hits' $\partial\Sigma$)

(Σ2) If Σ is unorientable and has p boundary-components, then we have loops

$$A_1, \ldots, A_g, C_1, \ldots, C_p$$

generating $\pi_1(\Sigma, o)$ subject to the relation that

$$C_p \cdots C_1 A_g A_g \cdots A_1 A_1 \text{ gives the identity in } \pi_1(\Sigma, o).$$

The loops C_i are of the form $\overline{L}_i C'_i L_i$, as described for (Σ1).

5.2 The connection spaces $\mathcal{A}(\Theta)$ and $\mathcal{A}(\underline{c})$. We are interested in connections ω with specified values, or restrictions, on the values of the holonomies $h(C_i;\omega)$. For simplicity we shall work with simultaneous restrictions on all of the C_i; restrictions on only some of the C_i can be handled similarly. Thus we consider the spaces of connections:

$$\mathcal{A}(c_1,\ldots,c_p) \stackrel{\text{def}}{=} \{\omega \in \mathcal{A} : h(C_1;\omega) = c_1,\ldots,h(C_p;\omega) = c_p\}, \qquad (5.2.1)$$
$$\text{for any } (c_1,\ldots,c_p) \in G^p,$$

$$\mathcal{A}(\Theta) \stackrel{\text{def}}{=} \left\{\omega \in \mathcal{A} : \Big(h(C_1;\omega),\ldots,h(C_p;\omega)\Big) \in \Theta\right\}, \qquad (5.2.2)$$
$$\text{for any } G\text{-conjugation orbit } \Theta \text{ in } G^p.$$

Here, by *G-conjugation orbit* we mean an orbit of the conjugation action of G on G^p, i.e. a subset of the form $\{(xc_1x^{-1},\ldots,xc_px^{-1}) : x \in G\}$.

We shall use the notation

$$\underline{c} \stackrel{\text{def}}{=} (c_1,\ldots,c_p). \qquad (5.2.3)$$

5.3 The measures $\mu_T^{\underline{c}}$ and μ_T^{Θ}. Let T be a fixed positive real number. On $\mathcal{A}(\underline{c})/\mathcal{G}_o$ we are interested in the measure

$$d\mu_T^{\underline{c}}([\omega]) = \frac{1}{Z_T(\underline{c})} e^{-S_{\mathrm{YM}}(\omega)/T} \delta\Big(\big(h(C_1;\omega),\ldots,h(C_p;\omega)\big)\underline{c}^{-1}\Big)[\mathcal{D}\omega], \qquad (5.3.1)$$

while on $\mathcal{A}(\Theta)/\mathcal{G}$ we are interested in the measure

$$d\mu_T^{\Theta}([\omega]) = \frac{1}{Z_T(\Theta)} e^{-S_{\mathrm{YM}}(\omega)/T} \delta_\Theta\Big(\big(h(C_1;\omega),\ldots,h(C_p;\omega)\big)\Big)[\mathcal{D}\omega], \qquad (5.3.2)$$

wherein δ_Θ is specified in the manner explained after (2.7.5).

As in the case of the torus, we start with the Gaussian probability space

$$(\Omega_{\mathrm{disk}}, \mu_{\mathrm{T,disk}}) \text{ for the disk } D. \qquad (5.3.3)$$

Then we define

$$\Omega_{\underline{c}} \stackrel{\text{def}}{=} \begin{cases} \Omega_{\mathrm{disk}} \times G^{2g} & \text{if } \Sigma \text{ is orientable, i.e. satisfies } (\Sigma 1) \\ \Omega_{\mathrm{disk}} \times G^{g} & \text{if } \Sigma \text{ satisifies } (\Sigma 2) \end{cases}. \qquad (5.3.4)$$

Similarly,

$$\Omega_{\Theta} \stackrel{\text{def}}{=} \begin{cases} \Omega_{\mathrm{disk}} \times G^{2g} \times \Theta & \text{if } \Sigma \text{ satisfies } (\Sigma 1) \\ \Omega_{\mathrm{disk}} \times G^{g} \times \Theta & \text{if } \Sigma \text{ satisifies } (\Sigma 2) \end{cases}. \qquad (5.3.5)$$

We define the measure $\mu_{\underline{c}}$ to be $\mu_{\mathrm{T,disk}} \times$ (Haar on $G^{\sigma g}$) (the σ in the exponent being 2 if $(\Sigma 1)$ applies and 1 if $(\Sigma 2)$ applies) *conditioned* to satisfy the following constraint (recall that L_0 is the radial path from O to $x_0 = (1,0) \in \partial D$):

$$h(\overline{L}_0 \cdot \partial D \cdot L_0;\omega_1) = \begin{cases} c_p \cdots c_1 b_g^{-1} a_g^{-1} b_g a_g \cdots b_1^{-1} a_1^{-1} ba & \text{if } (\Sigma 1) \text{ applies} \\ c_p \cdots c_1 a_g^2 \cdots a_1^2 & \text{if } (\Sigma 2) \text{ applies} \end{cases}, \qquad (5.3.6)$$

where

$$\omega \stackrel{\text{def}}{=} \begin{cases} (\omega_1, a_1, b_1, \ldots, a_g, b_g) & \text{if } (\Sigma 1) \text{ applies} \\ (\omega_1, a_1, \ldots, a_g) & \text{if } (\Sigma 2) \text{ applies} \end{cases}. \tag{5.3.7}$$

Thus the constraint will hold $\mu_T^{\underline{c}}$-almost-everywhere.

To define μ_T^{Θ} we use the same constraint condition on the measure

$$\mu_{\mathrm{T,disk}} \times (\text{Haar on } G^{\sigma g}) \times (G\text{-invariant unit-mass on } \Theta) \tag{5.3.8}$$

wherein σ is as before. The space on which μ_T^{Θ} will sometimes be denoted $\overline{\mathcal{A}(\Theta)/\mathcal{G}}$.

Expectation values with respect to $\mu_T^{\underline{c}}$ and μ_T^{Θ} may be obtained in a way exactly analogous to that explained in Sect. 3.4 for the one-holed torus.

Stochastic holonomies with respect to $\mu_T^{\underline{c}}$ and μ_T^{Θ} are also defined exactly analogously to (4.3.1) in Sect. 4.3.

Carrying out essentially the same arguments as in Sect. 4 we obtain the following generalized version of Theorems 4.4. and 4.6.

5.4.

Theorem 4. *Let $\kappa_1, \ldots, \kappa_n$ be loops on Σ, all based at the point o, composed of oriented 1−simplices of a good triangulation S of Σ (of the type described in Sect. 4.1). Let f be a bounded measurable function on G^n. Consider any $\underline{c} = (c_1, \ldots, c_p) \in G^p$. Then*

$$\begin{aligned} &\int f\,(h(\kappa_1;\omega), \ldots, h(\kappa_n;\omega))\; d\mu_T^{\underline{c}}(\omega) \\ &= \frac{1}{N_T(\underline{c})} \int f\,(x(\kappa_1), \ldots, x(\kappa_m)) \prod_{i=1}^{p} \delta\left(x(C_i)c_i^{-1}\right) \\ &\prod_{j=1}^{m} Q_{T|\Delta_j|}\left(x(\partial\Delta_j)\right) dx_{e_1} \cdots dx_{e_M}, \end{aligned} \tag{5.4.4a}$$

where $\Delta_1, \ldots, \Delta_m$ are the positively oriented 2−simplices of S, $e_1, \ldots, e_M$ and their reverses are the distinct oriented 1−simplices of S, $Q_t(\cdot)$ is the heat kernel on G normalized to $\int_G Q_t(x)\,dx = 1$, dx being unit-mass Haar measure on G, and

$$N_T(\underline{c}) \stackrel{\text{def}}{=} \begin{cases} \int_{G^{2g}} Q_{T|\Sigma|}(c_p \cdots c_1 b_g^{-1} a_g^{-1} b_g a_g \cdots b_1^{-1} a_1^{-1} b_1 a_1)\, da_1 \cdots db_g & \textit{if } (\Sigma 1) \textit{ applies} \\ \int_{G^g} Q_{T|\Sigma|}(c_p \cdots c_1 a_g^2 \cdots a_1^2)\, da_1 \cdots da_g & \textit{if } (\Sigma 2) \textit{ applies} \end{cases}.$$

The meaning of the δ functions in (5.4a) is that, for each C_i, for some arbitrary bond e_j lying on $\partial\Sigma$ and appearing in the loop C_i, the variable x_{e_j} should be replaced by the value which makes $x(C_i) = c_i$ and dx_{e_j} should be dropped from the integration.

For μ_T^{Θ}, we have:

$$\begin{aligned} &\int f\,(h(\kappa_1;\omega), \ldots, h(\kappa_n;\omega))\; d\mu_T^{\Theta}(\omega) \\ &= \frac{1}{N_T(\Theta)} \int f\,(x(\kappa_1), \ldots, x(\kappa_n)) \prod_{i=1}^{p} \delta_{\Theta}\left(x(C_i)c_i^{-1}\right) d_{\Theta}c_i \\ &\prod_{j=1}^{m} Q_{T|\Delta_j|}\left(x(\partial\Delta_j)\right) dx_{e_1} \cdots dx_{e_M}. \end{aligned}$$

The meaning of the delta-function δ_Θ and the integrator $d_\Theta c$ are explained in Sect. 2.7. The normalizer $N_T(\Theta)$ is $\int_{G^p} N_T(\underline{c})\, d_\Theta \underline{c}$.

The normalizer $N_T(\underline{c})$ has appeared in the appendix of [Se2] as a topologically-invariant function associated to the system of loops $C_1, \ldots, C_p$ on Σ. The functions $N_T(\Theta)$ have appeared in [Wi1].

As in Sect. 4, the loop expectation value formulas (5.4a, b) are invariant under area-preserving homeomorphisms of Σ which fix the loops C_i pointwise. However, for $p > 1$, the analog of Theorem 4.7 need not hold.

6. Spaces of Connections and 2–Forms on Them

In this section we describe the surface, spaces of connections, and 2–forms which we shall be working with in Sects. 6–9. Some of the notation and definitions are lifted directly from [KS2,3].

6.1 The surface Σ, group G, the connection spaces $\mathcal{A}$, $\mathcal{A}(\Theta)$, $\mathcal{M}(\Theta)$. Henceforth, Σ will denote a compact connected oriented 2–dimensional manifold of genus $g \geq 1$, with a connected non-empty boundary $\partial\Sigma$. In Sects. 8 and 9, Σ will be taken to be also a Riemannian manifold. We shall use a fixed basepoint o in Σ, and piecewise smooth loops $A_1, B_1, \ldots, A_g, B_g, C : [0,1] \to \Sigma$, all based at a point $o \in \Sigma$, which generate the fundamental group $\pi_1(\Sigma, o)$, subject to the relation that $C\overline{B}_g\overline{A}_g B_g A_g \cdots\cdots \overline{B}_1\overline{A}_1 B_1 A_1$ is homotopic to the constant loop at o, wherein we denote by $\overline{X}$ the reverse of any path X. The loop C is of the form $\overline{L}C^*L$, where C^* is a simple loop around the boundary $\partial\Sigma$ and L is a path from o to the initial point of C^*; the loops A_i and B_i are also of the form $A_i = \overline{L}A_i^*L$ and $B_i = \overline{L}B_i^*L$. A detailed description of these loops is given in [KS3: Sect. 6.1] (and in a wider setting in [Se2]), but we shall not need such details here.

We work with a principal G-bundle $\pi : P \to \Sigma$, where the gauge group G is now assumed to be compact, connected, and semisimple. The semisimplicity hypothesis is not of essential significance but we impose it to focus on the more significant issues.

Since Σ has boundary, the bundle P is trivial and so there is a smooth section $s : \Sigma \to P$. Connections and other forms on P can be pulled down to Σ by using s. In this way we can and will identify the space $\mathcal{A}$ of all connections on P with the space of all smooth $\underline{g}$-valued 1-forms over Σ; thus we take

$$\mathcal{A} = \text{space of all smooth } \underline{g}\text{-valued 1-forms on } \Sigma\,. \tag{6.1.1}$$

The section s is used only for convenience and the constructions (such as symplectic forms) and results we discuss are independent of the choice of s.

A connection $\omega \in \mathcal{A}$ is flat if its curvature is zero, i.e. if $d\omega + \frac{1}{2}[\omega, \omega] = 0$. We will be interested in the set of all flat connections :

$$\mathcal{A}^0 \stackrel{\text{def}}{=} \{\omega : \omega \in \mathcal{A} \text{ and } \omega \text{ is flat}\}. \tag{6.1.2}$$

We work with a fixed basepoint $u \in \pi^{-1}(o)$, where o is the basepoint on Σ. If κ is a piecewise smooth loop on Σ based at o, then, as explained in Sect. 2.2, $h(\kappa; \omega)$ denotes the *holonomy* of ω around κ, with u as initial point.

We shall denote by Θ a conjugacy class in the group G; we shall work with the spaces

$$\mathcal{A}(\Theta) = \{\omega \in \mathcal{A} : h(C;\omega) \in \Theta\} \text{ and } \mathcal{A}^0(\Theta) = \{\omega \in \mathcal{A}^0 : h(C;\omega) \in \Theta\}. \tag{6.1.3}$$

The group $\mathcal{G}$ (Sect. 2.3) of automorphisms of P can be identified, via the section s, with the set of all smooth maps $\Sigma \to G$; the group structure is now simply pointwise multiplication. The subgroup $\mathcal{G}_o$ now consists of those $\phi \in \Sigma$ for which $\phi(o) = e$. The action of $\mathcal{G}$ on $\mathcal{A}$ is given by:

$$\mathcal{A} \times \mathcal{G} \to \mathcal{A} : (A, \phi) \mapsto A \cdot \phi \stackrel{\text{def}}{=} \mathrm{Ad}(\phi^{-1})A + \phi^{-1}d\phi. \tag{6.1.4}$$

This action carries the sets $\mathcal{A}(\Theta)$ and $\mathcal{A}^0(\Theta)$ into themselves, and we have the moduli space

$$\mathcal{M}(\Theta) = \mathcal{A}^0(\Theta)/\mathcal{G}. \tag{6.1.5}$$

6.2 The 2–forms Ω and Ω_Θ. We are interested in a certain 2–form Ω_Θ on $\mathcal{A}(\Theta)$ which was introduced in [KS3]. The definition is:

$$\Omega_\Theta \stackrel{\text{def}}{=} \Omega + \Omega_{\text{ex}} + \Omega_\nu. \tag{6.2.1}$$

We shall explain now the definitions of the 2–forms Ω, Ω_{ex} and Ω_ν. The standard 2–form Ω on $\mathcal{A}$ (as described in [AB]) is given by:

$$\Omega(A, B) \stackrel{\text{def}}{=} \int_\Sigma \langle A \wedge B \rangle. \tag{6.2.2}$$

where A and B are $\underline{g}$–valued 1–forms on Σ and the product $\langle A \wedge B \rangle$ is the 2–form on Σ given by $\langle A \wedge B\rangle(X, Y) = \langle A(X), B(Y)\rangle_{\underline{g}} - \langle A(Y), B(X)\rangle_{\underline{g}}$.

Recall, from section 6.1, the loop C, part of which goes around $\partial\Sigma$. Let A be a tangent vector to $\mathcal{A}$ (i.e. A is a $\underline{g}$–valued 1–form on Σ), and define

$$\alpha : [0, 1] \to \underline{g} : t \mapsto \alpha(t) = -\int_0^t \mathrm{Ad}(h_s^{-1})A\big(C'(s)\big)\, ds, \tag{6.2.3}$$

where $s \mapsto h_s$ describes parallel transport along C: $h'(s)h(s)^{-1} = -A\left(C'(s)\right)$ with $h(0) = e$. It is known (and readily verifiable) that $\alpha(t)$ is the variation in h_t corresponding to the variation A in the connection ω. Define $\beta : [0, 1] \to G$ similarly with respect to a tangent vector B to $\mathcal{A}$. The 2–form Ω_{ex} is defined by:

$$\Omega_{\text{ex}}(A, B) \stackrel{\text{def}}{=} -\frac{1}{2}\int_0^1 \int_0^1 \epsilon_{st}\langle \alpha'(s), \beta'(t)\rangle\, ds\, dt, \tag{6.2.4}$$

where

$$\epsilon_{st} = \begin{cases} 1 & \text{if } s \le t; \\ -1 & \text{if } s > t. \end{cases}$$

Next, the 2–form Ω_ν on $\mathcal{A}(\Theta)$ is defined by:

$$\Omega_\nu\big|_\omega(A, B) = -\frac{1}{2}\langle \alpha(1), \left(\mathrm{Ad}(c) - 1\right)^{-1}\beta(1)\rangle + \frac{1}{2}\langle \beta(1), \left(\mathrm{Ad}(c) - 1\right)^{-1}\alpha(1)\rangle, \tag{6.2.5}$$

where $c = h(C; \omega)$, C being the loop for $\partial\Sigma$ as described in Sect. 6.1, and α, β ar as in (6.2.3). The terms involving $\left(\mathrm{Ad}(c) - 1\right)^{-1}$ are understood by setting $(\mathrm{Ad}(c) - 1)^{-1}$ to be 0 on $\ker[\mathrm{Ad}(c) - 1] = \left[(\mathrm{Ad}(c) - 1)(\underline{g})\right]^\perp$. Thus all terms on the right of the definition (6.2.1) of Ω_Θ have been specified.

6.3 The results of [KS3]. For ease of reference, we shall record here certain facts relating to $\mathcal{M}(\Theta)$ and Ω_Θ, including most of the results of [KS3].

(i) *The 2–form Ω_Θ is equivariant under the action of $\mathcal{G}$. Let $A, B \in T_\omega\mathcal{A}(\Theta)$ (an element of $T_\omega\mathcal{A}(\Theta)$ is, by definition, of the form $\partial\omega_t/\partial t|_{t=0}$ where $t \mapsto \omega_t \in \mathcal{A}(\Theta)$ is such that $(t,p) \mapsto \omega_t(p)$ is smooth and $\omega_0 = \omega$) and suppose that A or B is tangent to the $\mathcal{G}$–orbit through ω, in the sense that it is equal to $\partial(\omega\cdot\phi_t)/\partial t|_{t=0}$ for some path $t \mapsto \phi_t \in \mathcal{G}$ with $(t,p) \mapsto \phi_t(p)$ smooth and $\phi_0 = id$. Then $\Omega_\Theta(A, B) = 0$* (Proposition 5.1 in [KS3].)

(ii) *The holonomy representation*

$$\mathcal{H} : \omega \mapsto \left(h(A_1;\omega), h(B_1;\omega), \dots, h(A_g;\omega), h(B_g;\omega), h(C;\omega)\right) \in G^{2g} \times \Theta, \tag{6.3.1}$$

induces a bijection $\mathcal{A}(\Theta)/\mathcal{G}_o \to \Pi^{-1}(e)$, where $\mathcal{G}_o$ is the subgroup of $\mathcal{G}$ consisting of all ϕ with $\phi(o) = e$, and

$$\Pi : G^{2g} \times \Theta \to G : (a_1, b_1, \dots, a_g, b_g, c) \mapsto cb_g^{-1}a_g^{-1}b_ga_g \cdots b_1^{-1}a_1^{-1}b_1a_1. \tag{6.3.2}$$

The group G acts on the right of $\Pi^{-1}(e)$ by conjugation, and the holonomy mapping $\mathcal{H}$ is $\mathcal{G} - G$ equivariant with respect to the homomorphism $\mathcal{G} \to G : \phi \mapsto \phi(o)$, and induces a bijection of the full moduli spaces

$$\mathcal{M}(\Theta) \to \Pi^{-1}(e)/G. \tag{6.3.3}$$

(iii) *There is a dense open subset $\mathcal{D}$ of G such that for every conjugacy class Θ passing through any point in $\mathcal{D}$, the set $\Pi^{-1}(e)$ is a smooth submanifold of $G^{2g} \times \Theta$.*

(iv) *There is a 2–form $\overline{\Omega}_{o,\Theta}$ on $G^{2g} \times \Theta$ whose restriction to $\Pi^{-1}(e)$ pulls back to Ω_Θ by the holonomy map $\mathcal{H}$.*

(v) *If $\alpha \in \Pi^{-1}(e)$ and $A, B \in T_\alpha\Pi^{-1}(e)$ (this being taken, by definition, to be* $\ker d\Pi_\alpha$*) then $\overline{\Omega}_{o,\Theta}(A, B)$ is 0 if A or B is tangent to the G–orbit through α; moreover, $\overline{\Omega}_{o,\Theta}$ is G-equivariant. In this sense $\overline{\Omega}_{o,\Theta}$ descends to a 2–form, also denoted $\overline{\Omega}_\Theta$, on $\Pi^{-1}(e)/G$.* (Theorem 3.6 in [KS3])

(vi) *The 2–form $\overline{\Omega}_\Theta$ is closed (i.e. $\overline{\Omega}_{o,\Theta}$ on $\Pi^{-1}(e)$ is closed); it is non-degenerate on the smooth part of $\Pi^{-1}(e)/G$ for all Θ passing through a dense open subset of a certain neighborhood of the identity in G* (Theorem 4.1 in [KS3]). *By the 'generic (or smooth) part of $\Pi^{-1}(e)/G$' we mean the subset of points corresponding to the points on $\Pi^{-1}(e)$ where the derivative $d\Pi$ is surjective and where the isotropy of the G–action is minimal within any component.* (Sect. 7.4 below gives a more detailed explanation of 'generic part' of $\mathcal{M}(\Theta)$)

(vii) Formula specifying $\overline{\Omega}_\Theta$: *Let $(\alpha H^{(1)}, cC^{(1)}), (\alpha H^{(2)}, cC^{(2)}) \in T_{(\alpha,c)}\left(G^{2g} \times \Theta\right)$; then*

$$\begin{aligned}
&\overline{\Omega}_{o,\Theta}\left((\alpha H^{(1)}, cC^{(1)}), (\alpha H^{(1)}, cC^{(1)})\right)\\
&= \frac{1}{2}\sum_{1\le i,k\le 4g} \epsilon_{ik}\langle \mathrm{Ad}\,(\alpha_{i-1}\cdots\alpha_1)^{-1}H_i^{(1)}, \mathrm{Ad}\,(\alpha_{k-1}\cdots\alpha_1)^{-1}H_k^{(1)}\rangle_{\underline{g}}\\
&-\frac{1}{2}\langle\left(\mathrm{Ad}\,c^{-1} - 1\right)^{-1}C^{(1)}, \left(\mathrm{Ad}\,c^{-1} - \mathrm{Ad}\,c\right)\left(\mathrm{Ad}\,c^{-1} - 1\right)^{-1}C^{(2)}\rangle_{\underline{g}},
\end{aligned} \tag{6.3.4}$$

where

$$\epsilon_{ik} = \begin{cases} 1 & \text{if } i < k \\ 0 & \text{if } i = k \\ -1 & \text{if } i > k \end{cases},$$

and we have used the convenient if unusual notation of setting

$$\alpha = (\alpha_1, \alpha_2, \alpha_5, \alpha_6, \ldots, \alpha_{4g-3}, \alpha_{4g-2}) \tag{6.3.5}$$

and

$$\alpha_{j+2} = \alpha_j^{-1} \text{ for all } j \in J \stackrel{\text{def}}{=} \{1, 2, 5, 6, \ldots, 4g-3, 4g-2\}, \tag{6.3.6}$$

and, analogously, $H^{(r)} = (H_1^{(r)}, H_2^{(r)}, H_5^{(r)}, H_6^{(r)}, \ldots, H_{4g-3}^{(r)}, H_{4g-2}^{(r)})$, *and*

$$H_{j+2} = -\mathrm{Ad}\,(\alpha_j)H_j \text{ for all } j \in J. \tag{6.3.7}$$

In (6.3.4), *the terms* $(\mathrm{Ad}\,c^{-1} - 1)^{-1}$ *are meaningful because* $cC^{(1)}, cC^{(2)} \in T_c\Theta$ *and, as is readily seen,* $c^{-1}T_c\Theta = (\mathrm{Ad}\,c^{-1} - 1)(\underline{g})$.

Note that (6.3.4) specifies the 2–form $\overline{\Omega}_{o,\Theta}$ on all of $G^{2g} \times \Theta$; the 2–form $\overline{\Omega}_\Theta$ is obtained from its restriction to $\Pi^{-1}(e)$.

6.4 The sense in which $\mathcal{H}$ induces $\overline{\Omega}_\Theta$ from Ω_Θ. We wish to make a technical remark here concerning Sect. 6.3(iv). A precise statement of 6.3(iv) is as follows. Let $(\alpha, c) \in \Pi^{-1}(e) \subset G^{2g} \times \Theta$, and consider vectors $v_1, v_2 \in T_{(\alpha,c)}\left(G^{2g} \times \Theta\right)$ which are tangent to smooth paths lying entirely on $\Pi^{-1}(e)$. It has been shown in Sects. 4.1–4.4 of [KS2] that:

(a) there are paths $[0, 1] \to \mathcal{A}^0(\Theta) : t \mapsto \omega_t^i$, with $i = 1, 2$, such that $(t, p) \mapsto \omega_t^i(p)$ is smooth and $\mathcal{H}\left(\omega_t^i\right)$ is initially tangent to v_i, for $i = 1, 2$.

(b) $\Omega_\Theta(V_1, V_2) = \overline{\Omega}_{o,\Theta}(v_1, v_2)$, where $V_i = \partial\omega_i^t/\partial t|_{t=0}$, for $i = 1, 2$.

In view of Sect. 6.3(i), it is then reasonable to say that $\overline{\Omega}_{o,\Theta}$ gives the 2–form on $\Pi^{-1}(e) \simeq \mathcal{A}(\Theta)/\mathcal{G}_o$ (as in Sect. 6.3(ii)) induced by Ω_Θ, and hence that the 2–form $\overline{\Omega}_\Theta$ for $\mathcal{M}(\Theta)$ is induced by Ω_Θ. However, to make this a strictly logical conclusion one should verify that if $A, B \in T_\omega\mathcal{A}^0(\Theta)$ are such that $\mathcal{H}'(\omega)A$ or $\mathcal{H}'(\omega)B$ (the derivatives being pointwise partial derivatives, for instance $\mathcal{H}'(\omega)A = \partial\mathcal{H}(\omega_t)/\partial t|_{t=0}$ if $t \mapsto \omega_t \in \mathcal{A}^0(\Theta)$ is such that $(t, p) \mapsto \omega_t(p)$ is smooth and $\partial\omega_t(p)/\partial t|_{t=0} = A$) is 0 then $\Omega_\Theta(A, B) = 0$. For the case of closed surfaces this has been proven in Theorem 5.9.1(i) of [Se6] by showing that $\mathcal{H}'(\omega)A$ is 0 if and only if A is tangent to the $\mathcal{G}_o$–orbit through ω. We expect that this result (and, with minor modifications, its proof) holds for surfaces with boundary. A full treatment of this issue in the setting of compact surfaces with any number of boundary components is postponed to a future investigation. For the purposes of this paper it will suffice to take the relationship of Ω_Θ and $\overline{\Omega}_\Theta$ as specified above by (a) and (b).

7. Properties of $\overline{\Omega}_\Theta$, a Determinant Identity, and Non-Degeneracy for $\overline{\Omega}_\Theta$

The goal of this section is to prove that $\overline{\Omega}_\Theta$ is non-degenerate. For this we shall first prove a determinant identity ((7.8.1)) which will be useful again in Sects. 8 and 9.

7.1 Notation. We record here some notation some of which has already been used in Sect. 6. We work with a conjugacy class Θ in G, and we use the map

$$\Pi : G^{2g} \times \Theta \to G : (a_1, b_1, ..., a_g, b_g, c) \mapsto c b_g^{-1} a_g^{-1} b_g a_g \cdots b_1^{-1} a_1^{-1} b_1 a_1 . \quad (7.1.1)$$

The indexing set

$$J \overset{\text{def}}{=} \{1, 2, 5, 6, ..., 4g-3, 4g-2\} \quad (7.1.2)$$

is convenient to label a typical point in $G^{2g} \times \Theta$ as

$$\alpha = \left(\{\alpha_j\}_{j \in J}, c\right) .$$

Thus, comparing with the usual notation $(a_1, b_1, ..., a_g, b_g, c)$ we have set $\alpha_1 = a_1$, $\alpha_2 = b_1, ..., \alpha_{4g-3} = a_g$, $\alpha_{4g-2} = b_g$. As seen in the expression for $\overline{\Omega}_{o,\Theta}$ in (6.3.4), it is also convenient to introduce

$$\alpha_3 = \alpha_1^{-1}, \alpha_4 = \alpha_2^{-1}, \ldots, \alpha_{4g-1} = \alpha_{4g-3}^{-1}, \alpha_{4g} = \alpha_{4g-2}^{-1} . \quad (7.1.3)$$

Correspondingly, a vector in $T_{(\alpha,c)}(G^{2g} \times \Theta)$ has the form $\left(\{\alpha_j H_j\}_{j \in J}, cC\right)$, with $C \in c^{-1} T_c \Theta \subset \underline{g}$ (not to be confused with the loop C itself), and we set

$$H_{j+2} = -(\mathrm{Ad}\, \alpha_j) H_j \text{ for } j \in J . \quad (7.1.4)$$

It may be convenient to view the above notation as an expression of the imbedding

$$G^{2g} \times \Theta \to G^{4g} \times \Theta : (a_1, b_1, ..., a_g, b_g, c) \mapsto (a_1, b_1, a_1^{-1}, b_1^{-1}, ..., a_g, b_g, a_g^{-1}, b_g^{-1}, c). \quad (7.1.5)$$

The derivative of Π at a point $(\alpha, c) \in G^{2g} \times \Theta$ will be taken, by means of left translation, as a map $T_{(\alpha,c)}\left(G^{2g} \times \Theta\right) \to \underline{g}$; it is given by

$$d\Pi_{(\alpha,c)}(\alpha H, cC) = \sum_{i=1}^{4g} f_{i-1}^{-1} H_i + f_{4g}^{-1} C, \quad (7.1.6)$$

wherein

$$f_i = \mathrm{Ad}\,(\alpha_i \cdots \alpha_1). \quad (7.1.7)$$

Alternatively,

$$d\Pi_{(\alpha,c)}(\alpha H, cC) = \sum_{j \in J} (f_{j-1}^{-1} - f_{j+2}^{-1}) H_j + f_{4g}^{-1} C. \quad (7.1.8)$$

Here we have used the fact that for any $j \in J$,

$$\mathrm{Ad}(f_{j+1}^{-1}) H_{j+2} = -\mathrm{Ad}(f_{j+2}^{-1}) H_j . \quad (7.1.9)$$

This form is useful for determining the adjoint $d\Pi^*_{(\alpha,c)}$, which, again by left translations, we will take as a map $\underline{g} \to (\underline{g}^{2g}) \oplus \left(c^{-1} T_c \Theta\right)$; it is given by:

$$d\Pi^*_{(\alpha,c)} X = \left(\{(f_{j-1} - f_{j+2}) X\}_{j \in J}, \mathrm{pr}_{2g+1} f_{4g} X\right) , \quad (7.1.10)$$

where pr_{2g+1} is the orthogonal projection $\underline{g} \to c^{-1} T_c \Theta$. For the orbit map

$$\gamma_{(\alpha,c)} : G \to G^{2g} \times \Theta : x \mapsto \gamma_{(\alpha,c)}(x) = \left(\{x \alpha_j x^{-1}\}_{j \in J}, x c x^{-1}\right) \quad (7.1.11)$$

the derivative is the map $\underline{g} \to \underline{g}^{2g} \oplus \left(c^{-1}T_c\Theta\right)$ given by

$$\gamma'_{(\alpha,c)}(\alpha H, cC) = \left(\left\{\left(\operatorname{Ad}\alpha_j^{-1} - 1\right) H_j\right\}_{j\in J}, \left(\operatorname{Ad}c^{-1} - 1\right) C\right). \tag{7.1.12}$$

Here C is of course an element of $c^{-1}T_c\Theta \subset \underline{g}$, and is not the loop C of Sect. 6.1.

7.2 Interpretation of maps of the form $(\operatorname{Ad}c - 1)^{-1}$. We record some simple facts, conventions, and observations which will be useful later in computations. For $c \in G$, let

$$Z_c \stackrel{\text{def}}{=} \{X \in \underline{g} : (\operatorname{Ad}c)X = X\}. \tag{7.2.1}$$

Since

$$(\operatorname{Ad}c - 1)^* = \operatorname{Ad}c^{-1} - 1, \tag{7.2.2}$$

it follows that

$$\left(\operatorname{Ad}c^{\pm 1} - 1\right)(\underline{g}) = Z_c^{\perp} = c^{-1}T_c\Theta \tag{7.2.3}$$

and $(\operatorname{Ad}c^{\pm1} - 1)$ map $Z_c^{\perp}$ into itself isomorphically; by $\left(\operatorname{Ad}c^{\pm1} - 1\right)^{-1}$ we shall mean the inverse of this map. We have encountered and shall encounter again the composite $(\operatorname{Ad}c^{-1} - \operatorname{Ad}c)(\operatorname{Ad}c^{-1} - 1)^{-1}$. Observe that

$$\operatorname{Ad}c^{-1} - \operatorname{Ad}c = -(\operatorname{Ad}c - 1)(1 + \operatorname{Ad}c^{-1}) = (1 + \operatorname{Ad}c)(\operatorname{Ad}c^{-1} - 1). \tag{7.2.4}$$

Thus $(\operatorname{Ad}c^{-1} - \operatorname{Ad}c)$ maps $\underline{g}$ into $Z_c^{\perp}$, and so $(\operatorname{Ad}c^{-1} - \operatorname{Ad}c)(\operatorname{Ad}c^{-1} - 1)^{-1}$ maps $Z_c^{\perp}$ into itself. Splitting any $X \in \underline{g}$ into a component in Z_c and one in $Z_c^{\perp}$ shows that

$$(\operatorname{Ad}c^{-1} - \operatorname{Ad}c)(\operatorname{Ad}c^{-1} - 1)^{-1}(\operatorname{Ad}c^{-1} - 1)X = (\operatorname{Ad}c^{-1} - \operatorname{Ad}c)X. \tag{7.2.5}$$

From (7.2.4) we have, for any $C^{(1)} \in Z_c^{\perp}$ and $X \in \underline{g}$,

$$\begin{aligned}
&\left\langle \left(\operatorname{Ad}c^{-1} - 1\right)^{-1} C^{(1)}, \left(\operatorname{Ad}c^{-1} - \operatorname{Ad}c\right) X\right\rangle_{\underline{g}} \\
&= -\left\langle C^{(1)}, \left(1 + \operatorname{Ad}c^{-1}\right) X\right\rangle_{\underline{g}} \\
&= \left\langle C^{(1)}, (\operatorname{Ad}c - 1)^{-1}\left(\operatorname{Ad}c^{-1} - \operatorname{Ad}c\right) X\right\rangle,
\end{aligned} \tag{7.2.6}$$

where in the second equality we used the hypothesis that $C^{(1)} \in Z_c^{\perp}$ and the fact (following from (7.2.4)) that

$$(\operatorname{Ad}c - 1)^{-1}\left(\operatorname{Ad}c^{-1} - \operatorname{Ad}c\right) X = -\left(1 + \operatorname{Ad}c^{-1}\right) X + \text{an element of } Z_c.$$

The following result (similar to a result discussed in [KS1] for closed surfaces) says that the map Π, introduced in (6.3.2), plays a role somewhat analogous to that of a moment map.

7.3.

Lemma 7.1. *Let* $(\alpha, c) \in \Pi^{-1}(e) \subset G^{2g} \times \Theta$, $(\alpha H^{(1)}, cC) \in T_{(\alpha,c)}(G^{2g} \times \Theta)$, *and* $X \in \underline{g}$. *Then*

$$\overline{\Omega}_{o,\Theta}\left(\left(\alpha H^{(1)}, cC^{(1)}\right), \gamma'_{(\alpha,c)}(X)\right) = \left\langle d\Pi_{(\alpha,c)}(\alpha H^{(1)}, cC^{(1)}), \left(\frac{1 + Ad\,c}{2}\right) X\right\rangle_{\underline{g}}. \tag{7.3.1}$$

Proof. Recall that $\alpha = \{\alpha_j\}_{j\in J}$, and $\alpha_{j+2} = \alpha_j^{-1}$, for every $j \in J$, and $f_i \stackrel{\text{def}}{=} \mathrm{Ad}(\alpha_i \cdots \alpha_1)$ for every $i \in \{1, ..., 4g\}$. Recall also the expression (7.1.12) for $\gamma'_{(\alpha,c)}$. Then the expression for $\overline{\Omega}_{o,\Theta}$ in (6.3.4) gives:

$$\begin{aligned}
&\overline{\Omega}_{o,\Theta}\left((\alpha H^{(1)}, cC^{(1)}), \gamma'_{(\alpha,c)}(X)\right)\\
&\stackrel{(7.2.5)}{=} \frac{1}{2}\sum_{i,k=1}^{4g} \epsilon_{ik}\left\langle f_{i-1}^{-1}H_i^{(1)}, f_{k-1}^{-1}\left(\mathrm{Ad}\,\alpha_k^{-1} - 1\right)X\right\rangle\\
&-\frac{1}{2}\left\langle\left(\mathrm{Ad}\,c^{-1} - 1\right)^{-1}C^{(1)}, \left(\mathrm{Ad}\,c^{-1} - \mathrm{Ad}\,c\right)X\right\rangle\\
&\stackrel{(7.2.6)}{=} \frac{1}{2}\sum_{i=1}^{4g}\left\langle f_{i-1}^{-1}H_i^{(1)}, \{(1 - f_{i-1}^{-1}) + (f_{4g}^{-1} - f_i^{-1})\}X\right\rangle\\
&+\frac{1}{2}\left\langle C^{(1)}, \left(1 + \mathrm{Ad}\,c^{-1}\right)X\right\rangle\\
&\stackrel{(7.1.9)}{=} \frac{1}{2}\sum_{j\in J}\left\langle H_j^{(1)}, \left[f_{j-1}\left(1 + f_{4g}^{-1} - f_{j-1}^{-1} - f_j^{-1}\right)\right.\right.\\
&\left.\left.-f_{j+2}\left(1 + f_{4g}^{-1} - f_{j+1}^{-1} - f_{j+2}^{-1}\right)\right]X\right\rangle\\
&+\frac{1}{2}\left\langle(\mathrm{Ad}\,c)C^{(1)}, (1 + \mathrm{Ad}\,c)X\right\rangle\\
&= \sum_{j\in J}\left\langle H_j^{(1)}, (f_{j-1} - f_{j+2})\left(\frac{1 + f_{4g}^{-1}}{2}\right)X\right\rangle + \left\langle(\mathrm{Ad}\,c)C^{(1)}, \left(\frac{1 + \mathrm{Ad}\,c}{2}\right)X\right\rangle\\
&\stackrel{(7.1.8)}{=} \left\langle d\Pi_{(\alpha,c)}(\alpha H^{(1)}, cC^{(1)}), \left(\frac{1 + \mathrm{Ad}\,c}{2}\right)X\right\rangle,
\end{aligned}$$

where the last line is obtained from the hypothesis that $(\alpha, c) \in \Pi^{-1}(e)$ which implies that $(\mathrm{Ad}\,c)f_{4g} = 1$. $\square$

7.4 The generic stratum of $\mathcal{M}(\Theta)$. The moduli space $\mathcal{M}(\Theta)$ is, in general, not a manifold. For the purposes of this paper we shall define the tangent space $T_{(\alpha,c)}\Pi^{-1}(e)$ to be $\ker d\Pi_{(\alpha,c)}$; this could contain vectors which are not tangent to any paths in $\Pi^{-1}(e)$. By a k–form on $\Pi^{-1}(e)$ we mean the restriction of a k–form on $G^{2g} \times \Theta$ to (the tangent spaces of) $\Pi^{-1}(e)$. A G–equivariant k–form η on $\Pi^{-1}(e)$ for which $\eta(v_1, ..., v_k) = 0$ whenever any v_i is in the image of the derivative of the orbit map of the conjugation action of G will be taken to be a k–form $\overline{\eta}$ over $\mathcal{M}(\Theta)$. Such a form $\overline{\eta}$ is *closed* if it arises from a form η' on $G^{2g} \times \Theta$ for which the restriction of $d\eta'$ to the tangent spaces of $\Pi^{-1}(e)$ is 0.

A simple argument based on the expressions for $d\Pi^*_{(\alpha,c)}$ and $\gamma'_{(\alpha,c)}$ given in (7.1.10) and (7.1.12) shows that if $(\alpha, c) \in \Pi^{-1}(e)$ then

$$\ker d\Pi^*_{(\alpha,c)} = \ker \gamma'_{(\alpha,c)}. \tag{7.4.1}$$

It has been shown by means of Sard's theorem in Sect. 3.2 of [KS4] that there is a dense open subset of G such that if Θ includes a point in this set then Π is a submersion at every point on $\Pi^{-1}(e)$ and thus $\Pi^{-1}(e)$ is a smooth submanifold of $G^{2g} \times \Theta$. We shall focus on such Θ, to be called generic Θ.

For generic Θ, we see by (7.4.1) that the conjugation action of G on $\Pi^{-1}(e)$ is locally free. Standard results of transformation group theory ([Bre]) imply that in each connected component of $\Pi^{-1}(e)$ there is a dense open subset such that the quotient of this under the action of G is a smooth manifold and the quotient map is a submersion. It follows that there is a dense open subset of $\mathcal{M}(\Theta)$which is a manifold, to be called the *generic part* (or stratum) of $\mathcal{M}(\Theta)$, of dimension $(2g-2)\dim G + \dim(\Theta)$. The quotient map $\Pi^{-1}(e) \to \mathcal{M}(\Theta)$ over this generic part is a submersion.

It will be convenient to define the following 'tangent space':

$$T_p\mathcal{M}(\Theta) \stackrel{\text{def}}{=} \left(d\Pi_p^*(\underline{g}) + \gamma_p'(\underline{g})\right)^{\perp} \subset T_p\Pi^{-1}(e). \tag{7.4.2}$$

The subspace in (7.4.2) is equivariant under the conjugation action of G.

If p projects to a point $\overline{p}$ in the generic part of $\mathcal{M}(\Theta)$ then the derivative at p of the quotient map sets up

$$\text{a } G\text{-invariant isomorphism } q_* \text{ of } T_p\mathcal{M}(\Theta) \text{ onto } T_{\overline{p}}\mathcal{M}(\Theta). \tag{7.4.3}$$

Smooth G-equivariant k-forms on $\Pi^{-1}(e)$ which vanish on the directions $\gamma_p'(\underline{g})$ correspond one-to-one by q^* to smooth $k-$forms on the generic part of $\mathcal{M}(\Theta)$.

$$\begin{gathered}\text{We equip } T_{\overline{p}}\mathcal{M}(\Theta) \text{ with the inner-product which makes}\\ \text{the isomorphism } q_* \text{ in (7.4.2) an isometry.}\end{gathered} \tag{7.4.4}$$

Some of the discussion and results below carry over to more general Θ, and we expect that suitable sharper versions of the results below exist for all conjugacy classes Θ. Towards this, note that for general Θ we may define $q^*\overline{\Omega}_\Theta$ at $p \in \Pi^{-1}(e)$ to mean the restriction of $\overline{\Omega}_{o,\Theta}$ to $T_p\mathcal{M}(\Theta)$, and define $\det \overline{\Omega}_{o,\Theta}$ to be the determinant of the restriction of $\overline{\Omega}_{o,\Theta}$ on $T_p\mathcal{M}(\Theta)$ with respect to any orthonormal basis in $T_p\mathcal{M}(\Theta)$ (Sect. 7.6 contains more on such determinants).

The case where Θ consists of one point (this being any point lying necessarily in the center of G) becomes essentially identical to the theory for closed surfaces, i.e. the theory covered by [KS1].

7.5 .

Lemma 7.2. *Split $T_{(\alpha,c)}(G^{2g} \times \Theta)$ as a direct sum of orthogonal pieces*

$$T_{(\alpha,c)}(G^{2g} \times \Theta) = d\Pi^*_{(\alpha,c)}(\underline{g}) \oplus \gamma'_{(\alpha,c)}(\underline{g}) \oplus T_{(\alpha,c)}\mathcal{M}(\Theta). \tag{7.5.1}$$

Then, with respect to this decomposition, $\overline{\Omega}_{o,\Theta}$ has the 'matrix form'

$$\begin{bmatrix} * & Q & * \\ -Q^t & 0 & 0 \\ * & 0 & q^*\overline{\Omega}_\Theta \end{bmatrix}, \tag{7.5.2}$$

where Q is the bilinear map $d\Pi^*_{(\alpha,c)}(\underline{g}) \times \gamma'_{(\alpha,c)}(\underline{g}) \to \mathbf{R}$ given by

$$Q\left(Y, \gamma'_{(\alpha,c)}X\right) = \left\langle d\Pi_{(\alpha,c)}Y\,,\ \frac{1+Ad\,c}{2}X\right\rangle \tag{7.5.3}$$

and Q^t is the bilinear map $\gamma'_{(\alpha,c)}(\underline{g}) \times d\Pi^*_{(\alpha,c)}(\underline{g}) \to \mathbf{R}$ given on $\left(\gamma'_{(\alpha,c)}X, Y\right)$ by the right side of (7.5.3).

Proof. First we observe that $(\alpha, c) \in \Pi^{-1}(e)$ implies that $\Pi \circ \gamma_{(\alpha,c)}(x) = e$ for every $x \in G$, and so $d\Pi^*_{(\alpha,c)}(\underline{g})$ and $\gamma'_{(\alpha,c)}(\underline{g})$ are indeed orthogonal. From (7.3.1) it follows that $\overline{\Omega}_{o,\Theta}\left(Y, \gamma'_{(\alpha,c)}X\right)$ is 0 when $Y \in \ker d\Pi_{(\alpha,c)}$; since $d\Pi^*_{(\alpha,c)}(\underline{g})^{\perp} = \ker d\Pi_{(\alpha,c)}$, this explains the two zeros in the second column. The top block Q in the second column is, by definition of the "matrix form",

$$Q(A, B) \stackrel{\text{def}}{=} \overline{\Omega}_{\Theta}(A, B)$$

and so the expression (7.5.3) for Q is simply a restatement of (7.3.1). The second block in the first column now follows by skew-symmetry of $\overline{\Omega}_{o,\Theta}$. The bottom corner block follows from the fact that $\overline{\Omega}_{\Theta}$ is, by definition, the image of $\overline{\Omega}_{o,\Theta}$ under the isomorphism q_* of (7.4.4). □

7.6 Determinants. Let $A : V \to W$ be a linear map between finite-dimensional inner-product spaces. If $\ker A \neq \{0\}$, or if $V = \{0\}$, then we define $\det(A) = 0$. If A ($\neq 0$) is an isomorphism onto its image $A(V)$, then by $\det A$ we shall mean the determinant of a matrix of A relative to orthonormal bases in V and $A(V)$. Thus $\det A$ is determined up to sign, but is otherwise independent of the choice of bases. Consideration of matrices shows that $\det\left(A|(\ker A)^{\perp}\right) = \det\left(A^*|A(V)\right)$. If A is an isomorphism onto W, and if $B : W \to Z$ is a linear map into a finite dimensional inner-product space Z, then $\det(BA) = \det(B)\det(A)$.

If $P : V \times W \to \mathbf{R}$ is a bilinear form, where V and W are finite-dimensional inner-product spaces of equal dimension, then by $\det P$ (determined up to sign) we shall mean the determinant of the matrix $\left[P(v_i, w_j)\right]$, where $\{v_i\}$ and $\{w_j\}$ are orthonormal bases in V and W, respectively; $\det P$ is taken to be 0 if V and W are 0–dimensional. The determinant $\det \overline{\Omega}_{\Theta}$, which we shall use below, is to be understood in this sense.

7.7.

Lemma 7.3. *Let $(\alpha, c) \in \Pi^{-1}(e)$. Denote by $\langle d\Pi \otimes d\Pi\rangle$ the bilinear form on $T_{(\alpha,c)}(G^{2g} \times \Theta)$ defined by*

$$\langle d\Pi \otimes d\Pi\rangle(X, Y) = \langle d\Pi_{(\alpha,c)}X, d\Pi_{(\alpha,c)}Y\rangle_{\underline{g}}, \tag{7.7.1}$$

and let $\langle d\Pi \otimes (Ad\, c)\mathrm{pr}_{2g+1}\rangle$ be the bilinear form on $T_{(\alpha,c)}(G^{2g} \times \Theta)$ given by

$$\langle d\Pi \otimes (Ad\, c)\mathrm{pr}_{2g+1}\rangle(X, Y) = \langle d\Pi_{(\alpha,c)}X, (Ad\, c)\mathrm{pr}_{2g+1}Y\rangle, \tag{7.7.2}$$

where pr_{2g+1} is the projection of $T_{(\alpha,c)}\left(G^{2g} \times \Theta\right)$ on the last factor $c^{-1}T_c\Theta$. Then, assuming that c is not in the center of G, (i.e. Θ consists of more than one point)

$$\begin{aligned}\det&\left(\overline{\Omega}_{o,\Theta} - \frac{1}{2}\langle d\Pi \otimes d\Pi\rangle + \frac{1}{2}\langle d\Pi \otimes (Ad\, c)\mathrm{pr}_{2g+1}\rangle\right)\\ &= \det\left(-\frac{1}{2}(Ad\, c - 1)^{-1}(Ad\, c^{-1} - Ad\, c)(Ad\, c^{-1} - 1)^{-1}\right),\end{aligned} \tag{7.7.3}$$

where $(Ad\, c^{-1} - 1)^{-1}$ and $(Ad\, c - 1)^{-1}$ are taken as maps from $(Ad\, c^{-1} - 1)(\underline{g})$ into itself.

Proof. Let $(\alpha, c) \in \Pi^{-1}(e)$, and consider vectors $(\alpha H^{(1)}, cC^{(1)}), (\alpha H^{(2)}, cC^{(2)}) \in T_{(\alpha,c)}(G^{2g} \times \Theta)$. Recall the expression for $\overline{\Omega}_{o,\Theta}$ in (6.3.4):

$$\begin{aligned}
&\overline{\Omega}_{o,\Theta}\left((\alpha H^{(1)}, cC^{(1)}), (\alpha H^{(2)}, cC^{(2)})\right) \\
&= \frac{1}{2}\sum_{i,k=1}^{4g} \epsilon_{ik}\left\langle f_{i-1}^{-1} H_i^{(1)}, f_{k-1}^{-1} H_k^{(2)}\right\rangle \\
&-\frac{1}{2}\Big\langle \left(\mathrm{Ad}\, c^{-1} - 1\right)^{-1} C^{(1)}, \\
&\left(\mathrm{Ad}\, c^{-1} - \mathrm{Ad}\, c\right)\left(\mathrm{Ad}\, c^{-1} - 1\right)^{-1} C^{(2)}\Big\rangle,
\end{aligned} \tag{7.7.4a}$$

where

$$f_i = \mathrm{Ad}(\alpha_i \cdots \alpha_1),$$
$$(\alpha_1, \ldots, \alpha_{4g}) = (a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1}),$$
$$\alpha = (a_1, b_1, \ldots, a_g, b_g) = \{\alpha_j\}_{j\in J},$$
$$J = \{1, 2, 5, 6, \ldots, 4g-3, 4g-2\},$$
$$H_{j+2}^{(r)} = -\left(\mathrm{Ad}\, \alpha_j\right) H_j^{(r)} \text{ for every } j \in J,$$

all as explained in Sect. 7.1. From these we have

$$\overline{\Omega}_{o,\Theta}\left((\alpha H^{(1)}, cC^{(1)}), (\alpha H^{(2)}, cC^{(2)})\right) = \sum_{j,k\in J}\langle H_j^{(1)}, \Omega_{jk} H_k^{(2)}\rangle + \langle C^{(1)}, \Omega_{2g+1\,2g+1} C^{(2)}\rangle, \tag{7.7.4b}$$

wherein (upon using the relationship (7.1.9) between H_{j+2} and H_j in the expression for $\overline{\Omega}_{o,\Theta}$ in (7.7.4*a*))

$$\Omega_{jk} = \frac{1}{2}\left[f_{j-1}\left(\epsilon_{jk} f_{k-1}^{-1} - \epsilon_{j\,k+2} f_{k+2}^{-1}\right) - f_{j+2}\left(\epsilon_{j+2\,k} f_{k-1}^{-1} - \epsilon_{jk} f_{k+2}^{-1}\right)\right] \tag{7.7.5a}$$

and, using the second equality in (7.2.6),

$$\Omega_{2g+1\,2g+1} = -\frac{1}{2}(\mathrm{Ad}\, c - 1)^{-1}(\mathrm{Ad}\, c^{-1} - \mathrm{Ad}\, c)(\mathrm{Ad}\, c^{-1} - 1)^{-1}. \tag{7.7.5b}$$

Recall that

$$J \stackrel{\text{def}}{=} \{1, 2, 5, 6, \ldots, 4g-3, 4g-2\},$$

so that $J \cup (J+2) = \{1, 2, \ldots, 4g\}$.

So, from (7.7.5*a*), for $j, k \in J$ with $j + 2 < k$,

$$\Omega_{jk} = \frac{1}{2}(f_{j-1} - f_{j+2})(f_{k-1}^{-1} - f_{k+2}^{-1}). \tag{7.7.6}$$

On the other hand, from the expression for $d\Pi_{(\alpha,c)}$ in (7.1.8) we have:

$$\langle d\Pi \otimes d\Pi\rangle\left((\alpha H^{(1)}, cC^{(1)}), (\alpha H^{(2)}, cC^{(2)})\right) \tag{7.7.7}$$

$$\stackrel{(7.1.8)}{=} \sum_{j,k\in J}\left\langle (f_{j-1}^{-1} - f_{j+2}^{-1}) H_j^{(1)}, (f_{k-1}^{-1} - f_{k+2}^{-1}) H_k^{(2)}\right\rangle \tag{7.7.8}$$

$$+\sum_{j\in J}\left\langle (f_{j-1}^{-1} - f_{j+2}^{-1}) H_j^{(1)}, (\mathrm{Ad}\, c) C^{(2)}\right\rangle$$

$$+\left\langle (\mathrm{Ad}\,c)C^{(1)}, \left(f_{k-1}^{-1} - f_{k+2}^{-1}\right) H_k^{(2)} \right\rangle + \langle C^{(1)}, C^{(2)} \rangle \tag{7.7.9}$$

and

$$\begin{aligned} &\langle d\Pi \otimes (\mathrm{Ad}\,c)\mathrm{pr}_{2g+1} \rangle \left((\alpha H^{(1)}, cC^{(1)}), (\alpha H^{(2)}, cC^{(2)}\right) \\ &\overset{(7.1.8)}{=} \left\langle \sum_{j\in J} (f_{j-1}^{-1} - f_{j+2}^{-1}) H_j^{(1)}, (\mathrm{Ad}\,c)C^{(2)} \right\rangle + \langle C^{(1)}, C^{(2)} \rangle. \end{aligned} \tag{7.7.10}$$

Thus, setting

$$\Omega' \overset{\text{def}}{=} \overline{\Omega}_{o,\Theta} - \frac{1}{2}\langle d\Pi \otimes d\Pi \rangle + \frac{1}{2}\langle d\Pi \otimes (\mathrm{Ad}\,c)\mathrm{pr}_{2g+1} \rangle, \tag{7.7.9}$$

we can write

$$\begin{aligned} &\Omega' \left((\alpha H^{(1)}, cC^{(1)}), (\alpha H^{(2)}, cC^{(2)})\right) \\ &= \sum_{j,k\in J} \left\langle H_j^{(1)}, \Omega'_{jk} H_k^{(2)} \right\rangle + \sum_{j\in J} \left\langle H_j^{(1)}, \Omega'_{j\,2g+1} C^{(2)} \right\rangle \\ &+ \sum_{k\in J} \left\langle C^{(2)}, \Omega'_{2g+1\,k} H_k^{(2)} \right\rangle + \left\langle C^{(1)}, \Omega'_{2g+1\,2g+1} C^{(2)} \right\rangle, \end{aligned} \tag{7.7.10}$$

where

$$\Omega'_{2g+1\,2g+1} = -\frac{1}{2}(\mathrm{Ad}\,c - 1)^{-1}(\mathrm{Ad}\,c^{-1} - \mathrm{Ad}\,c)(\mathrm{Ad}\,c^{-1} - 1)^{-1} \tag{7.7.11}$$

and, from (7.7.6), (7.7.7) and (7.7.8),

$$\Omega'_{jk} = \tfrac{1}{2}(f_{j-1} - f_{j+2})(f_{k-1}^{-1} - f_{k+2}^{-1}) - \tfrac{1}{2}(f_{j-1} - f_{j+2})(f_{k-1}^{-1} - f_{k+2}^{-1}) = 0 \quad \text{for } j, k \in J \text{ with } j+2 < k \tag{7.7.12}$$

and from (7.7.4*b*), (7.7.7) and (7.7.8),

$$\Omega'_{j\,2g+1} = 0 - \frac{1}{2}(f_{j-1} - f_{j+2})\mathrm{Ad}\,c + \frac{1}{2}(f_{j-1} - f_{j+2})\mathrm{Ad}\,c = 0 \text{ for } j \in J \tag{(7.7.13)}$$

and, similarly,

$$\Omega'_{2g+1\,k} = -\tfrac{1}{2}\mathrm{Ad}\,c^{-1}\left(f_{k-1}^{-1} - f_{k+2}^{-1}\right) \text{ for } k \in J. \tag{7.7.14}$$

Thus the 'matrix' $[\Omega'_{jk}]_{j,k\in J}$ for Ω' has an upper triangular form:

$$\Omega' = \begin{bmatrix} D_1 & 0 & 0 & \cdots & \cdots & 0 \\ * & D_5 & 0 & \cdots & \cdots & 0 \\ * & * & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & D_j & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & * & D_{4g-3} & 0 \\ * & * & * & * & * & \Omega'_{2g+1\,2g+1} \end{bmatrix}, \tag{7.7.15}$$

where the diagonal entry D_j, for $j, j+1 \in J$, is given from (7.7.5*a*), (7.7.7) and (7.7.9), by

$$
\begin{aligned}
D_j &\stackrel{\text{def}}{=} \begin{bmatrix} \Omega'_{jj} & \Omega'_{j\,j+1} \\ \Omega'_{j+1\,j} & \Omega'_{j+1\,j+1} \end{bmatrix} \\
&= \begin{bmatrix} f_{j+2}f_{j-1}^{-1} - 1 & f_{j+2}f_j^{-1} \\ -f_j f_{j-1}^{-1} + f_{j+3}f_{j-1}^{-1} - f_{j+3}f_{j+2}^{-1} & f_{j+3}f_j^{-1} - 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{Ad}\,(\alpha_j^{-1}\alpha_{j+1}\alpha_j) - 1 & \mathrm{Ad}\,(\alpha_j^{-1}\alpha_{j+1}) \\ -\mathrm{Ad}\,\alpha_j + \mathrm{Ad}\,(\alpha_{j+1}^{-1}\alpha_j^{-1}\alpha_{j+1}\alpha_j) - \mathrm{Ad}\,\alpha_{j+1}^{-1} & \mathrm{Ad}\,(\alpha_{j+1}^{-1}\alpha_j^{-1}\alpha_{j+1}) - 1 \end{bmatrix}.
\end{aligned} \tag{7.7.16}
$$

This factorizes (as observed in [KS1]) as

$$
D_j = \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & a^{-1} - 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b - 1 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \tag{7.7.17}
$$

where $a = \mathrm{Ad}\,\alpha_j$ and $b = \mathrm{Ad}\,\alpha_{j+1}$. This implies that $\det(D_j) = 1$. Combining this observation with the form for Ω' given in (7.7.15), we have:

$$
\det \Omega' = \det \Omega'_{2g+1\,2g+1} \prod_{j \in J'} \det D_j = \det \Omega'_{2g+1\,2g+1}, \tag{7.7.18}
$$

where $J' = \{j \in J : j+1 \in J\} = \{1, 5, \ldots, 4g-3\}$. Recalling the expression for $\Omega'_{2g+1\,2g+1}$ given in (7.7.11), the proof is complete. $\square$

7.8.

Lemma 7.4. *Let $(\alpha, c) \in \Pi^{-1}(e) \subset G^{2g} \times \Theta$, and suppose that $\Pi : G^{2g} \times \Theta \to G$ is a submersion at (α, c). Then*

$$
\det \overline{\Omega}_\Theta = \left(\frac{\det \gamma'_{(\alpha,c)}}{\det d\Pi^*_{(\alpha,c)}} \right)^2 \det \left[(1 - Ad\,c)^{-1} \right], \tag{7.8.1}
$$

where $\det \left[(1 - Ad\,c)^{-1} \right]$ *is the determinant of the map*

$$
(1 - Ad\,c)^{-1} : Z_c^\perp \to Z_c^\perp
$$

with $Z_c^\perp$ being the subspace $(\ker(Ad\,c - 1))^\perp = (1 - Ad\,c)(\underline{g})$.

In particular, $\overline{\Omega}_\Theta$ is non-degenerate on the smooth part of $\Pi^{-1}(e)/G$.

Proof. Recall from Lemma 7.5 the splitting $T_{(\alpha,c)}(G^{2g} \times \Theta)$ as a direct sum of orthogonal pieces

$$
T_{(\alpha,c)}(G^{2g} \times \Theta) = d\Pi^*_{(\alpha,c)}(\underline{g}) \oplus \gamma'_{(\alpha,c)}(\underline{g}) \oplus T_{(\alpha,c)}\mathcal{M}(\Theta) \tag{7.8.2}
$$

and the corresponding 'matrix form' of $\overline{\Omega}_{o,\Theta}$ given by

$$
\begin{bmatrix} * & Q & * \\ -Q^t & 0 & 0 \\ * & 0 & q^*\overline{\Omega}_\Theta \end{bmatrix}, \tag{7.8.3}
$$

where

$$
Q\left(Y, \gamma'_{(\alpha,c)}X\right) = \left\langle d\Pi_{(\alpha,c)}Y\,,\, \frac{1 + \mathrm{Ad}\,c}{2} X \right\rangle \tag{7.8.4}
$$

for every $Y \in d\Pi^*_{(\alpha,c)}(\underline{g})$, and $q_* : T_{(\alpha,c)}\mathcal{M}(\Theta) \to T_{\overline{(\alpha,c)}}\mathcal{M}(\Theta)$ is as explained in Sect. 7.4.

Note that in the decomposition (7.8.2), the sum of the second and third summands is $\ker d\Pi_{(\alpha,c)}$. Using this we see that the 'matrix' for $\frac{1}{2}\langle d\Pi \otimes d\Pi\rangle$ at (α, c) has the form

$$\begin{bmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.8.5}$$

and the matrix for $\frac{1}{2}\langle d\Pi \otimes (\operatorname{Ad} c)\mathrm{pr}_{2g+1}\rangle$ at (α, c) has the form

$$\begin{bmatrix} * & Q_1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{7.8.6}$$

where Q_1 is the bilinear map $d\Pi^*_{(\alpha,c)}(\underline{g}) \times \gamma'_{(\alpha,c)}(\underline{g}) \to \mathbf{R}$ given by

$$Q_1\left(Y, \gamma'_{(\alpha,c)}X\right) = \frac{1}{2}\langle d\Pi_{(\alpha,c)}Y,\ (\operatorname{Ad} c)(\operatorname{Ad} c^{-1} - 1)X\rangle. \tag{7.8.7}$$

Recall the bilinear form Ω' on $T_{(\alpha,c)}(G^{2g} \times \Theta)$ given in (7.7.9):

$$\Omega' \stackrel{\text{def}}{=} \overline{\Omega}_{o,\Theta} - \frac{1}{2}\langle d\Pi \otimes d\Pi\rangle + \frac{1}{2}\langle d\Pi \otimes (\operatorname{Ad} c)\mathrm{pr}_{2g+1}\rangle.$$

From the preceding observations we see that, relative to the splitting of $T_{(\alpha,c)}(G^{2g} \times \Theta)$ as in (7.8.1), Ω' has the matrix form:

$$\begin{bmatrix} * & Q_2 & * \\ -Q^t & 0 & 0 \\ 0 & * & q^*\overline{\Omega}_\Theta \end{bmatrix}, \tag{7.8.8}$$

where $Q_2 = Q + Q_1$ is the bilinear map $d\Pi^*_{(\alpha,c)}(\underline{g}) \times \gamma'_{(\alpha,c)}(\underline{g}) \to \mathbf{R}$ given by

$$Q_2\left(Y, \gamma'_{(\alpha,c)}X\right) = \langle d\Pi_{(\alpha,c)}Y, X\rangle = \left\langle Y\ ,\ d\Pi^*_{(\alpha,c)}X\right\rangle \tag{7.8.9}$$

and $-Q^t$ is the bilinear map $\gamma'_{(\alpha,c)}(\underline{g}) \times d\Pi^*_{(\alpha,c)}(\underline{g}) \to \mathbf{R}$ given by

$$-Q^t\left(\gamma'_{(\alpha,c)}X, Y\right) = -\left\langle d\Pi_{(\alpha,c)}Y, \frac{1 + \operatorname{Ad} c}{2}X\right\rangle.$$

From (7.8.8) we have, as usual not worrying about signs,

$$\det \Omega' = \det Q_2 \det(-Q^t) \det \overline{\Omega}_\Theta,$$

and the expressions for Q_2 and $-Q^t$ then imply that

$$\det \Omega' = \left\{\frac{\det d\Pi^*_{(\alpha,c)}}{\det \gamma'_{(\alpha,c)}} \cdot \frac{\det d\Pi^*_{(\alpha,c)}}{\det \gamma'_{(\alpha,c)}} \det\left(-\frac{1 + \operatorname{Ad} c}{2}\right)\right\} \det \overline{\Omega}_\Theta. \tag{7.8.10}$$

Now decomposing $\underline{g}$ as the orthogonal sum of $Z_c = \ker(\operatorname{Ad} c - 1)$ and $Z_c^\perp$, we have

$$\det\left(\frac{1 + \operatorname{Ad} c}{2}\right) = \det\left(\frac{1 + \operatorname{Ad} c}{2}\Big| Z_c^\perp \to Z_c^\perp\right). \tag{7.8.11}$$

Combining the expression for $\det \Omega'$ given in (7.8.10) with that obtained earlier in (7.7.3), and using (7.8.11), we obtain

$$\begin{aligned} \det \overline{\Omega}_\Theta &= \left(\frac{\det \gamma'_{(\alpha,c)}}{\det d\Pi^*_{(\alpha,c)}}\right)^2 \frac{\det\left[(\mathrm{Ad}\,c-1)^{-1}(\mathrm{Ad}\,c-\mathrm{Ad}\,c^{-1})(\mathrm{Ad}\,c^{-1}-1)^{-1}\right]}{\det\left((1+\mathrm{Ad}\,c)\middle|Z_c^\perp\right)} \\ &\overset{(7.2.4)}{=} \left(\frac{\det \gamma'_{(\alpha,c)}}{\det d\Pi^*_{(\alpha,c)}}\right)^2 \frac{\det(1-\mathrm{Ad}\,c)^{-1}\det\left((1+\mathrm{Ad}\,c)\middle|Z_c^\perp\right)}{\det\left((1+\mathrm{Ad}\,c)\middle|Z_c^\perp\right)}, \end{aligned} \tag{7.8.12}$$

thus proving the determinant identity (7.8.1). □

Recall from Sect. 6.3(iii) that there is a dense open subset $\mathcal{D} \subset G$ such that for every conjugacy class Θ passing through any point of $\mathcal{D}$, Π is a submersion at every point of the level set $\Pi^{-1}(e)$ and thus $\Pi^{-1}(e)$ is a smooth submanifold of $G^{2g} \times \Theta$. In Sect. 7.4 we saw that $\overline{\Omega}_\Theta$ is a smooth 2–form on a subset (which we call the generic part) of $\mathcal{M}(\Theta)$ which is a smooth manifold; as noted in Sect. 6.3(vi), it has been proven in [KS2] that $\overline{\Omega}_\Theta$ is a closed 2–form on this generic part of $\mathcal{M}(\Theta)$. Combining this with Lemma 7.8 we obtain the following result.

7.9.

Theorem 5. *The 2–form $\overline{\Omega}_\Theta$ is a symplectic structure on the generic part of $\mathcal{M}(\Theta)$ for every generic conjugacy class Θ in G.*

8. Limiting Quantum Yang-Mills and Symplectic Volume

In this section Σ is equipped with a Riemannian metric; i.e. Σ is a compact connected smooth Riemannian manifold with one connected boundary component $\partial\Sigma$. We will prove that as $T \downarrow 0$ the measures μ_T^Θ (given heuristically by (2.7.5) and rigorously in Sect. 5.3) converge, in a sense specified below, to a volume measure on $\mathcal{M}(\Theta) = \mathcal{A}^0(\Theta)/\mathcal{G}$ which corresponds locally to the symplectic structure $\overline{\Omega}_\Theta$ on $\mathcal{M}(\Theta)$ (if λ is a symplectic 2–form on a 2d–dimensional space, then the corresponding volume form vol_λ is the exterior power $\wedge^{\mathrm{d}}\lambda/\mathrm{d}!$).

Recall that the measure μ_T^Θ, as constructed in Sect. 5.3, is a probability measure on a space $\overline{\mathcal{A}(\Theta)/\mathcal{G}_o}$. For each well-behaved loop κ on Σ based at o there is a random variable $\omega \mapsto h(\kappa;\omega) \in G$ which corresponds to the the holonomy of ω around κ. For our present purposes we shall not need any details of the definition of $\overline{\mathcal{A}(\Theta)/\mathcal{G}_o}$, nor of the stochastic random holonomies, nor of what 'well-behaved' loops mean. What we shall need is summarized in the following special case of Theorem 5.4, which may be taken as a specification of μ_T^Θ, thus providing a choice of a starting point for our present discussions.

The triangulations of Σ that we use are assumed to be 'admissible' in the sense of Sect. 4.1. Alternatively, for the purposes of the present section, we may work with any arbitrary triangulation of Σ and consider μ_T^Θ as being a measure defined by means of (8.1.1) below.

8.1 Loop expectation values for μ_T^{Θ}.
Let S be a (well-behaved) triangulation of Σ, with the basepoint o as a 0-simplex. Then for any loops $\kappa_1, \ldots, \kappa_n$ based at o and consisting of oriented 1-simplices of S, and any bounded measurable function f on G^n,

$$\int f\left(h(\kappa_1;\omega), \ldots, h(\kappa_n;\omega)\right) d\mu_T^{\Theta}(\omega)$$
$$= N_T(\Theta)^{-1} \int f\left(x(\kappa_1), \ldots, x(\kappa_n)\right) \delta\left(x(C)c^{-1}\right)$$
$$\prod_{j=1}^{m} Q_{T|\Delta_j|}\left(x(\partial\Delta_j)\right) dx_{e_1} \cdots dx_{e_M} d_{\Theta}c, \tag{8.1.1}$$

where the notation is as explained in Theorem 5.4. The normalizer $N_T(\Theta)$ is given by:

$$N_T(\Theta) = \int_{G^{2g}} Q_{T|\Sigma|}(cb_g^{-1}a_g^{-1}b_g a_g \cdots b_1^{-1}a_1^{-1}b_1 a_1)\, da\, db\, d_{\Theta}c. \tag{8.1.2}$$

By appropriate subdivision of S and by joining o to a vertex of each Δ_j by an appropriate curve l_j, each loop κ in S based at o can be expressed as a 'composite' of a sequence of the loops of the form $\bar{l}_j \cdot \Delta_j \cdot l_j$ and their reverses, and the loops A_i, B_i, C and their reverses (by 'composite' here we include the operation of successively dropping edges which are traversed in opposite directions consecutively in the usual composite of curves). This has been described in more detail in Sect. 4.2. Each $x(\kappa_i)$ in (8.1.1) is then a product of the $x(l_j)^{-1}x(\partial\Delta_j)^{\pm 1}x(l_j)$ and $x(A_i)^{\pm 1}$, $x(B_i)^{\pm 1}$, and $x(C)^{\pm 1}$. Thus

$$\text{we can express } f\left(x(\kappa_1), \ldots, x(\kappa_n)\right) \text{ as } F\left(y_{\Delta_1}, \ldots, y_{\Delta_m}, \{a_i, b_i\}, c\right) \tag{8.1.3}$$

for some function F, where

$$y_{\Delta_j} = x(l_j)^{-1}x(\partial\Delta_j)x(l_j), \tag{8.1.4}$$

$$a_i = x(A_i), b_i = x(B_i), c = x(C). \tag{8.1.5}$$

Conversely, given $y_{\Delta_1}, \ldots, y_{\Delta_m}$, $\{a_i, b_i\}$, c in G, satisfying

$$y_{\Delta_m} \cdots y_{\Delta_1} = cb_g^{-1}a_g^{-1}b_g a_g \cdots b_1^{-1}a_1^{-1}b_1 a_1.$$

there is an assignment $e \mapsto x_e$, with $x_{\bar{e}} = x_e^{-1}$ for every oriented $1-$simplex e of S, such that (8.1.4) and (8.1.5) hold.

With this change of variables it follows, as for (4.4.5),

$$\int f\left(h(\kappa_1;\omega), \ldots, h(\kappa_n;\omega)\right) d\mu_T^{\Theta}(\omega)$$
$$= \frac{1}{N_T(\Theta)} \int_{G^m \times G^{2g} \times \Theta} F\left(y_{\Delta_1}, \ldots, y_{\Delta_m}, \{a_i, b_i\}, c\right) \cdot$$
$$\cdot\, \delta\left(y_{\Delta_m} \cdots y_{\Delta_1}(cb_g^{-1}a_g^{-1}b_g a_g b_1^{-1}a_1^{-1}b_1 a_1)^{-1}\right) d_{\Theta}c$$
$$\prod_{i=1}^{g} da_i db_i \prod_{j=1}^{m} Q_{T|\Delta_j|}(y_{\Delta_j}) dy_{\Delta_j}, \tag{8.1.6}$$

where the $\delta(\cdot)$ term means that any one of the y_{Δ_j} may be replaced in the integrand by the value which makes the argument in $\delta(\cdot)$ equal to e, and the corresponding integration dy_{Δ_j}, along with the $\delta(\cdot)$–term, dropped from the integration.

Taking $T \downarrow 0$ we obtain the following lemma which is very close to Lemma 2.14 in [Se4]; since the sketch proof presented in [Se4] is sloppy enough to be viewed as incorrect we present a detailed argument here.

8.2 .

Lemma 8.5. *Let $\kappa_1, \dots, \kappa_n$ be loops based at o and composed of oriented 1–simplices of the triangulation S, let f be a continuous function on G^n, and let F be associated to f as described above in* (8.1.3). *Then*

$$\begin{aligned}\lim_{T\downarrow 0} &\int f(h(\kappa_1;\omega), \dots, h(\kappa_n;\omega))\, d\mu_T^{\Theta}(\omega) \\ &= \lim_{T\to 0} N_T(\Theta)^{-1} \int F(e, \dots, e, \{a_i, b_i\}, c) \\ &\cdot Q_{T|\Sigma|}\left(cb_g^{-1}a_g^{-1}b_g a_g \cdots b_1^{-1}a_1^{-1}b_1 a_1\right) da_1 db_1 \cdots da_g db_g d_\Theta c, \qquad (8.2.1)\end{aligned}$$

provided that the limit on the right side exists and provided that the limit

$$N_0(\Theta) \stackrel{\text{def}}{=} \lim_{T\to 0} \int_{G^{2g}} Q_{T|\Sigma|}(cb_g^{-1}a_g^{-1}b_g a_g \cdots b_1^{-1}a_1^{-1}b_1 a_1)\, da_1 db_1 \cdots da_g db_g d_\Theta c \qquad (8.2.2)$$

exists and is positive.

Proof. Let λ_T be the Borel measure on $G^m \times G^{2g} \times \Theta$ specified by requiring that for every continuous function h on $G^m \times G^{2g} \times \Theta$,

$$\begin{aligned}\int h\, d\lambda_T = &\int_{G^{m-1}\times G^{2g}\times\Theta} h\left(y_1, \dots y_{m-1}, y'_m, a_1, \dots, b_g, c\right) \cdot \\ &\cdot Q_{T|\Delta_m|}(y'_m) \left(\prod_{j=1}^{m-1} Q_{T|\Delta_j|}(y_j) dy_j\right) \left(\prod_{i=1}^{g} da_i db_i\right) d_\Theta c \qquad (8.2.3)\end{aligned}$$

wherein

$$y'_m = \{\Pi(a_1, \dots, b_g, c)\}\, (y_{m-1} \cdots y_1)^{-1} \qquad (8.2.4)$$

and, as usual,

$$\Pi(a_1, \dots, b_g, c) = cb_g^{-1}a_g^{-1}b_g a_g \cdots b_1^{-1}a_1^{-1}b_1 a_1. \qquad (8.2.5)$$

By a change-of-variables argument, the role of y'_m (and of Δ_m) can be replaced by that of y'_j (and of Δ_j) in the integration (8.2.3).

The loop expectation value formula (8.1.6) says that

$$\begin{aligned}&\int f(h(\kappa_1;\omega), \dots, h(\kappa_n;\omega))\, d\mu_T^{\Theta}(\omega) \\ &= \frac{1}{N_T(\Theta)} \int F(y_1, \dots, y_m, \{a_i, b_i\}, c)\, d\lambda_T(\{y_j\}, \{a_i, b_i\}, c). \qquad (8.2.6)\end{aligned}$$

So we estimate

$$\left| \int F\left(y_1, \dots y_m, \{a_i, b_i\}, c\right) d\lambda_T(\{y_j\}, \{a_i, b_i\}, c)\right.$$

$$
\begin{aligned}
&- \int F\left(e,\ldots,e,\{a_i,b_i\},c\right) d\lambda_T(\{y_j\},\{a_i,b_i\},c)\Big| \\
&\leq \sum_{j=1}^{m} \int \Big| F(e,\ldots,e,y_j,\ldots,y_m,\{a_i,b_i\},c) - \\
&F(e,\ldots,e,y_{j+1},\ldots,y_m,\{a_i,b_i\},c)\Big| \, d\lambda_T(\{y_j\},\{a_i,b_i\},c) \qquad (8.2.7)
\end{aligned}
$$

wherein the $j = m$ term is interpreted using $F(e,\ldots,e,\{a_i,b_i\},c)$ as the second term in the integrand.

Let $\epsilon > 0$. By (uniform) continuity of F, there is a neighborhood U_ϵ of e such that, for any $j \in \{1,\ldots,m\}$, if $y_j \in U_\epsilon$ then the integrand on the right side in (8.2.7) is less than ϵ. Thus, since the total mass of λ_T is $N_T(\Theta)$,

$$
\text{left side of (8.2.7)} \leq m\epsilon \cdot N_T(\Theta) + 2\|F\|_{\sup} \sum_{j=1}^{m} \lambda_T(S_{j,\epsilon}), \qquad (8.2.8)
$$

where

$$
S_{j,\epsilon} \stackrel{\text{def}}{=} \{(y_1,\ldots,y_m,\{a_i,b_i\},c) \in G^{m+2g} \times \Theta : y_j \notin U_\epsilon\}. \qquad (8.2.9)
$$

Using the convolution property $\int Q_t(ab)Q_s(b^{-1}c)\,da\,db = Q_{t+s}(ac)$, the conjugation invariance of Q_t, and $\int_G Q_t(x)\,dx = 1$, we have

$$
\begin{aligned}
\lambda_T(S_{j,\epsilon}) &= \int_{y \notin U_\epsilon} Q_{T|\Delta_j|}(y) Q_{T|\Sigma| - T|\Delta_j|}\left(\Pi(a_1,\ldots,b_g,c)y^{-1}\right) \, dy\,da_1 \ldots db_g d_\Theta c \\
&\leq \sup_{y \notin U_\epsilon} Q_{T|\Delta_j|}(y). \qquad (8.2.10)
\end{aligned}
$$

Since $\lim_{t \downarrow 0} \sup_{y \notin U} Q_t(y) = 0$ for any neighborhood U of e, we can divide (8.2.8) by $N_T(\Theta)$, let $T \downarrow 0$, and use the hypothesis that $N_0(\Theta)$ exists and is positive to conclude that (8.2.1) holds. □

For the following, recall from section 8.3 that for any Θ which passes through a certain dense open subset of G, the map $\Pi : G^{2g} \times \Theta \to G$ is a submersion at every point of $\Pi^{-1}(e)$.

8.3 .

Theorem 6. *Assume that*

(i) *Θ is such that Π is a submersion at every point of $\Pi^{-1}(e)$*

(ii) *$\Pi^{-1}(e)$ has a dense open subset $\Pi^{-1}(e)^0$ on which the isotropy of the conjugation action of G is $Z(G)$*

(iii) *$vol_{\overline{\Omega}_\Theta}\left(\mathcal{M}(\Theta)^0\right) < \infty$, where $\mathcal{M}(\Theta)^0$ is the projection of $\Pi^{-1}(e)^0$ onto $\mathcal{M}(\Theta)$ by the projection map $\Pi^{-1}(e) \to \Pi^{-1}(e)/G \simeq \mathcal{M}(\Theta)$ (as in (6.3.3)), and $vol_{\overline{\Omega}_\Theta}$ is the volume with respect to the symplectic structure $\overline{\Omega}_\Theta$ on $\mathcal{M}(\Theta)^0$.*

(iv) *$\kappa_1,\ldots,\kappa_n$ are loops on Σ based at o, as in section 8.1.*

Then for any continuous function f on G^n which is invariant under the conjugation action of G,

$$\lim_{T\downarrow 0}\int f\left(h(\kappa_1;\omega),\ldots,h(\kappa_n;\omega)\right)\,d\mu_T^{\Theta}(\omega)$$
$$=\frac{1}{vol_{\overline{\Omega}_\Theta}\left(\mathcal{M}(\Theta)^0\right)}\int_{\mathcal{M}(\Theta)^0} f\left(h(\kappa_1;\omega),\ldots,h(\kappa_n;\omega)\right)\,dvol_{\overline{\Omega}_\Theta}([\omega]) \quad (8.3.1)$$

Proof. In view of the limiting formulas given in Lemma 8.2, we shall focus on computing $\lim_{t\to 0}\int_{G^{2g}\times\Theta} H(a_1,\ldots,b_g,c)Q_t\left(\Pi(a_1,\ldots,b_g,c)\right)\,da_1\ldots db_g d_\Theta c$, for continuous functions H.

By the submersivity hypothesis, $\Pi^{-1}(e)$ is a submanifold of $G^{2g}\times\Theta$. Picking any point $x_*\in\Pi^{-1}(e)$ we have, again by submersivity of Π at x_*, a coordinate system χ on a neighborhood W of x_* in $G^{2g}\times\Theta$ and a coordinate system on a neighborhood U of $\Pi(x_*)$ in G such that $\Pi(W)=U$ and $\Pi|W$ corresponds, in the coordinates, to projection on the first $\dim(G)$ coordinates.

Let
$$V=(\Pi|W)^{-1}(e)=\Pi^{-1}(e)\cap W$$
Thus, taking the coordinate system χ such that $\chi(W)$ is a cube, there is a diffeomorphism
$$\Phi: U\times V\to W$$
such that $\Pi\circ\Phi: U\times V\to U$ is the projection on the first factor. Therefore, writing
$$w=\Phi(u,v),$$
the derivative $d\Phi_{(u,v)}:\underline{g}\oplus T_v\Pi^{-1}(e)\to T_{\Phi(u,v)}W = d\Pi_w^*(\underline{g})+\ker d\Pi_w$ can be expressed as a matrix of the form
$$\begin{array}{c|c} & \begin{array}{cc}\underline{g} & T_v\Pi^{-1}(e)\end{array}\\ \begin{array}{c} d\Pi_w^*(\underline{g})\\ \ker d\Pi_w\end{array} & \begin{bmatrix} B & 0\\ * & D_2\Phi_{(u,v)}\end{bmatrix}\end{array}, \quad (8.3.2)$$
where $D_2\Phi_{(u,v)}: T_v\Pi^{-1}(e)\to\ker d\Pi_{\Phi(u,v)}$ is the partial deriative of Φ in the second variable, and B^{-1} is the restriction of $d\Pi_w$ to $d\Pi_w^*(\underline{g})=(\ker d\Pi_w)^{\perp}$.

It should be noted that the diagonal blocks listed in (8.3.2) are indeed 'square blocks'. Moreover, $\det B^{-1}=\det d\Pi_w^*$. Consequently,
$$|\det d\Phi_{(u,v)}|=\frac{|\det D_2\Phi_{(u,v)}|}{|\det d\Pi^*_{\Phi(u,v)}|}. \quad (8.3.3)$$

Let H be a continuous function on $G^{2g}\times\Theta$ with compact support contained in W. For the following computations it is necessary to bear in mind that we use the unit-mass invariant measures on G and Θ, and these differ by constant volume factors from the respective Riemannian volume measures. Then, upon using (8.3.3), we have the following change-of-variables formula
$$\mathrm{vol}(G)^{2g}\mathrm{vol}(\Theta)\int_{G^{2g}\times\Theta} H\left(a_1,\ldots,b_g,c\right)Q_t\left(\Pi(a_1,\ldots,b_g,c)\right)\,da_1\ldots db_g d_\Theta c$$
$$=\mathrm{vol}(G)\int_{U\times V} H\left(\Phi(u,v)\right)Q_t(u)\frac{|\det D_2\Phi_{(u,v)}|}{|\det d\Pi^*_{\Phi(u,v)}|}\,du\,d\sigma(v), \quad (8.3.4)$$
where du is the (restriction to U of the) usual unit-mass Haar measure on G, $d\sigma(v)$ the Riemannian volume measure on $V\subset\Pi^{-1}(e)$, and vol denotes volume measured

with respect to the Riemannian metrics on G and Θ induced by $\langle\cdot,\cdot\rangle_{\underline{g}}$. Since H is continuous and has compact support contained in W, we can apply the heat-kernel property $\lim_{t\to 0}\int_G h(x)Q_t(x)\,dx = h(e)$ valid for all continuous functions h on G, to conclude that

$$\begin{aligned}&\lim_{t\downarrow 0}\int_{G^{2g}\times\Theta} H\left(a_1,\ldots,b_g,c\right) Q_t\left(\Pi(a_1,\ldots,b_g,c)\right)\,da_1\ldots db_g d_\Theta c\\ &= \mathrm{vol}(G)^{1-2g}\mathrm{vol}(\Theta)^{-1}\int_V H\left(\Phi(e,v)\right)\frac{|\det D_2\Phi_{(e,v)}|}{|\det d\Pi^*_{\Phi(e,v)}|}\,d\sigma(v).\end{aligned}\tag{8.3.5}$$

Since $|\det D_2\Phi_{(e,v)}|$ is the Jacobian, at v, of the map $V\to V : v\mapsto\Phi(e,v)$, we have

$$\begin{aligned}&\lim_{t\downarrow 0}\int_{G^{2g}\times\Theta} H\left(a_1,\ldots,b_g,c\right) Q_t\left(\Pi(a_1,\ldots,b_g,c)\right)\,da_1\ldots db_g d_\Theta c\\ &= \mathrm{vol}(G)^{1-2g}\mathrm{vol}(\Theta)^{-1}\int_V H(v)\frac{1}{|\det d\Pi^*_v|}\,d\sigma(v)\\ &= \mathrm{vol}(G)^{1-2g}\mathrm{vol}(\Theta)^{-1}\int_{\Pi^{-1}(e)} H(v)\frac{1}{|\det d\Pi^*_v|}\,d\sigma(v),\end{aligned}\tag{8.3.6a}$$

where in the last line we used again the fact that H is supported in W. By a standard partition-of-unity argument, we conclude that

$$\begin{aligned}&\lim_{t\downarrow 0}\int_{G^{2g}\times\Theta} H\left(a_1,\ldots,b_g,c\right) Q_t\left(\Pi(a_1,\ldots,b_g,c)\right)\,da_1\ldots db_g d_\Theta c\\ &= \mathrm{vol}(G)^{1-2g}\mathrm{vol}(\Theta)^{-1}\int_{\Pi^{-1}(e)} H(v)\frac{1}{|\det d\Pi^*_v|}\,d\sigma(v)\end{aligned}\tag{8.3.6b}$$

holds for every continuous function H on $G^{2g}\times\Theta$.

Applying this to the limiting formulas (8.2.1) and (8.2.2) we have

$$\begin{aligned}&\lim_{T\downarrow 0}\int f\left(h(\kappa_1;\omega),\ldots,h(\kappa_n;\omega)\right)\,d\mu_T^\Theta(\omega)\\ &= N_0(\Theta)^{-1}\mathrm{vol}(G)^{1-2g}\mathrm{vol}(\Theta)^{-1}\\ &\int_{\Pi^{-1}(e)} F\left(e,\ldots,e,\alpha,c\right)|\det(d\Pi_{(\alpha,c)})^*|^{-1}\,d\sigma(\alpha,c)\end{aligned}\tag{8.3.7}$$

with

$$N_0(\Theta) = \mathrm{vol}(G)^{1-2g}\mathrm{vol}(\Theta)^{-1}\int_{\Pi^{-1}(e)}|\det(d\Pi_{(\alpha,c)})^*|^{-1}\,d\sigma(\alpha,c)\tag{8.3.8}$$

and $d\sigma$ being the Riemannian volume-measure on $\Pi^{-1}(e)$; the limiting formula (8.3.7), and the existence of the right side of (8.3.7), is contingent upon $N_0(\Theta)$ being positive and finite. Positivity of $N_0(\Theta)$ is clear from (8.3.8) since $\Pi^{-1}(e)\neq\emptyset$; finiteness will be shown below in (8.3.13).

Let $\Pi^{-1}(e)^0$ be the dense open subset of $\Pi^{-1}(e)$ on which the isotropy of the G-action on the manifold is $Z(G)$. Then by standard facts from the theory of transformation groups (Sect. 16.4.1(i), Problem 16.10.1 and Problem 12.10.1(a) in [Die]), the projection $\Pi^{-1}(e)^0\to\mathcal{M}(\Theta)^0$ is a principal $G/Z(G)-$bundle over the manifold $\mathcal{M}(\Theta)^0$. Since f is $G-$invariant, it follows that so is F and hence the function $F(e,\ldots,e,(\cdot))|\Pi^{-1}(e)^0$

is constant on the fibers; we will sometimes view $F(e,\dots,e,(\cdot))$ as a function on the quotient $\Pi^{-1}(e)^0/G$. Writing $F(e,\dots,e,(\cdot))|\Pi^{-1}(e)^0$ as a sum of functions supported on subsets of $\Pi^{-1}(e)^0$ on which there are local-trivializations, it follows from (8.3.7) (see [KS1] or Lemma 5.8 of [Se5] for details of this argument) that

$$\begin{aligned}
&\lim_{T\downarrow 0}\int f\left(h(\kappa_1;\omega),\dots,h(\kappa_n;\omega)\right)\,d\mu_T^{\Theta}(\omega)\\
&= N_0(\Theta)^{-1}\frac{\mathrm{vol}(G)^{1-2g}}{\mathrm{vol}(\Theta)}\mathrm{vol}\left(G/Z(G)\right)\\
&\int_{\Pi^{-1}(e)^0/G} F\left(e,\dots,e,\alpha,c\right)\frac{|\det\gamma'_{(\alpha,c)}|}{|\det(d\Pi_{(\alpha,c)})^*|}\,d\overline{\sigma}(\alpha,c)
\end{aligned}\tag{8.3.9}$$

and

$$N_0(\Theta)=\frac{\mathrm{vol}(G)^{2-2g}}{\mathrm{vol}(\Theta)\cdot\#Z(G)}\int_{\Pi^{-1}(e)^0/G}\frac{|\det\gamma'_{(\alpha,c)}|}{|\det(d\Pi_{(\alpha,c)})^*|}\,d\overline{\sigma}(\alpha,c),\tag{8.3.10}$$

with $d\overline{\sigma}$ being the volume-measure on $\Pi^{-1}(e)^0/G$ corresponding to the Riemannian structure on $\Pi^{-1}(e)^0/G$ (this being as in (7.4.4)). Recall from Lemma 7.8 that

$$\frac{|\det\gamma'_{(\alpha,c)}|}{|\det d\Pi^*_{(\alpha,c)}|}=|\mathrm{Pf}(\overline{\Omega}_\Theta)|\left|\det(1-\mathrm{Ad}\,c)^{-1}\right|^{-1/2},\tag{8.3.11}$$

where the Pfaffian $|\mathrm{Pf}(\overline{\Omega}_\Theta)|$ is the square-root of $|\det\overline{\Omega}_\Theta|$. Furthermore, since $\mathrm{Pf}(\overline{\Omega}_\Theta)d\overline{\sigma}=d\mathrm{vol}_{\overline{\Omega}_\Theta}$, we see that the right side of (8.3.9) is given by

$$\begin{aligned}
&N_0(\Theta)^{-1}\left|\det(1-\mathrm{Ad}\,c)^{-1}\right|^{-1/2}\frac{\mathrm{vol}(G)^{1-2g}}{\mathrm{vol}(\Theta)}\mathrm{vol}\left(G/Z(G)\right)\cdot\\
&\cdot\int_{\Pi^{-1}(e)^0/G}F\left(e,\dots,e,\alpha,c\right)\,d\mathrm{vol}_{\overline{\Omega}_\Theta}\overline{(\alpha,c)}
\end{aligned}\tag{8.3.12}$$

and, recalling the identifications $\Pi^{-1}(e)/G\simeq\mathcal{M}(\Theta)$ and $\Pi^{-1}(e)^0/G\simeq\mathcal{M}(\Theta)^0$,

$$N_0(\Theta)=\left|\det(1-\mathrm{Ad}\,c)^{-1}\right|^{-1/2}\frac{\mathrm{vol}(G)^{2-2g}}{\mathrm{vol}(\Theta)\cdot\#Z(G)}\mathrm{vol}_{\overline{\Omega}_\Theta}\left(\mathcal{M}(\Theta)^0\right),\tag{8.3.13}$$

where c is any point in Θ. The hypothesis that $\mathrm{vol}_{\overline{\Omega}_\Theta}\left(\mathcal{M}(\Theta)^0\right)$ is finite now shows that $N_0(\Theta)$ is finite and this justifies (8.3.7) and hence also (8.3.9) and (8.3.12).

Now returning to the relationship between f and F explained in (8.1.3), (8.1.5) and in the remarks following (8.1.5), we have

$$f\left(h(\kappa_1;\omega),\dots,h(\kappa_n;\omega)\right)=F(e,\dots,e,a_1,\dots,b_g,c)\tag{8.3.14}$$

for any flat connection ω with $h(A_i;\omega)=a_i$, $h(B_i;\omega)=b_i$ and $h(C;\omega)=c$. Combining (8.3.9), (8.3.12) and (8.3.14), the proof is complete. □

9. The Case of $SU(2)$ Bundles

In this section we specialize the considerations of the preceding Sects. to the case $G = SU(2)$ and show that a sharp form of the semiclassical limit formula holds. We also determine the symplectic volume of $\mathcal{M}(\Theta)$ explicitly.

We shall exclude the case where Θ consists of one point (which must be one of the matrices $\pm I$). The excluded cases are essentially contained in the theory for $SU(2)$ and $SO(3)$ flat connections over closed surfaces, and this theory is treated fully in [Se5].

9.1 .

Theorem 7. *Let the group G be $SU(2)$, and let Θ be a conjugacy class containing more than one point (i.e. Θ is not the conjugacy class of I or of $-I$). Then:*
[i] *$\mathcal{M}(\Theta)$ is a smooth* (6g – 4)– *dimensional manifold which is* <u>*connected*</u>.
[ii] *the* 2*–form $\overline{\Omega}_\Theta$ on $\mathcal{M}(\Theta)$ is symplectic and the corresponding volume of $\mathcal{M}(\Theta)$ is*

$$vol_{\overline{\Omega}_\Theta}(\mathcal{M}(\Theta)) = \begin{cases} 2\pi(\pi-\theta_c)\left[\frac{vol(SU(2))}{2\pi^2}\right]^{\frac{2}{3}} & \text{if } g=1 \\ 4\pi\sin\theta_c\left[\frac{vol(SU(2))}{2\pi^2}\right]^{\frac{2}{3}} vol(SU(2))^{2g-2}\sum_{n=1}^{\infty}\frac{\chi_n(c)}{n^{2g-1}} & \text{if } g\geq 2 \end{cases}, \tag{9.1.1}$$

where χ_n is the character function of $SU(2)$ specified below in (9.1.7), *c is any point in the conjugacy class Θ, θ_c is the number in $(0,\pi)$ for which $\cos\theta_c = Tr(c)/2$, and $vol(SU(2))$ is the volume of $SU(2)$ with respect to the fixed metric $\langle\cdot,\cdot\rangle_{\underline{g}}$ on its Lie algebra.*
[iii] *if S be a triangulation of Σ, and $\kappa_1,\ldots,\kappa_n$ loops based at o made up of oriented* 1*– simplices of S as in Sect. then for any continuous conjugation-invariant function f on G^n,*

$$\lim_{T\downarrow 0}\int f(h(\kappa_1;\omega),\ldots,h(\kappa_n;\omega))\,d\mu_T^{\Theta}(\omega)$$
$$= \frac{1}{vol_{\overline{\Omega}_\Theta}(\mathcal{M}(\Theta))}\int_{\mathcal{M}(\Theta)} f(h(\kappa_1;\omega),\ldots,h(\kappa_n;\omega))\,dvol_{\overline{\Omega}_\Theta}([\omega]). \tag{9.1.2}$$

Proof. (i) In SU(2) any element not in the center lies in a unique maximal torus, and two elements commute if and only if both lie in the same maximal torus. This implies that the isotropy of the G conjugation action at any $(a_1,\ldots,b_g,c)\in\Pi^{-1}(I)\subset G^{2g}\times\Theta$ (with I being the identity matrix, and Θ being other than the one-point conjugacy class $\{I\}$) is $Z(G)=\{\pm I\}$. Consequently, from (7.4.1), Π is a submersion at every point of $\Pi^{-1}(I)$, and so $\Pi^{-1}(I)$ is a smooth submanifold of $G^{2g}\times\Theta$, of dimension $3(2g)+2-3 = 6g-1$. Since the isotropy group of the G action is $\{\pm I\}$ everywhere it follows, by standard facts from transformation group theory alluded to earlier ([Die]), that $\Pi^{-1}(I)/G\simeq\mathcal{M}(\Theta)$ is a smooth manifold of dimension 6g – 4, and $\Pi^{-1}(I)\to\Pi^{-1}(I)/G\simeq\mathcal{M}(\Theta)$ is a smooth principal $SU(2)/\{\pm I\}$–bundle. Let $(\alpha_1,c_1),(\alpha_2,c_2)\in\Pi^{-1}(I)$; in particular, $c_1,c_2\in\Theta$. Since $G=SU(2)$ is connected, there is a continuous path $[0,1]\to G: t\mapsto x_t$ such that $x_0=I$ and $x_1c_1x_1^{-1}=c_2$. Therefore $t\mapsto(x_t\alpha_1x_t^{-1},x_tc_1x_t^{-1})$ is a path in $\Pi^{-1}(I)$ from (α_1,c_1) to a point (α_2',c_2) where α_2' lies in $K^{-1}(c_2^{-1})$, where K is the product commutator map

$$K: G^{2g}\to G:(a_1,b_1,\ldots,a_g,b_g)\mapsto b_g^{-1}a_g^{-1}b_ga_g\cdots b_1^{-1}a_1^{-1}b_1a_1.$$

In Proposition 3.10 of [Se5], it was proven that the set $K^{-1}(c^{-1})$ is connected for every $c \in SU(2)$. Consequently, α_2' can be connected to α_2 by a continuous path lying in $K^{-1}(c_2^{-1})$. Combining all these observations we conclude that (α_1, c_1) can be connected to (α_2, c_2) by a continuous path in $\Pi^{-1}(I)$.

(ii) Recall from (8.3.13) that, since now $\mathcal{M}(\Theta)^0 = \mathcal{M}(\Theta)$,

$$\begin{aligned}&\lim_{t\to 0}\int_{G^{2g}\times\Theta} Q_t\left(cb_g^{-1}a_g^{-1}b_ga_g\cdots b_1^{-1}a_1^{-1}b_1a_1\right)\,da_1\cdots db_g d_\Theta c\\ &= \left|\det(1-\operatorname{Ad}c)^{-1}\right|^{-1/2}\frac{\operatorname{vol}(G)^{2-2g}}{\operatorname{vol}(\Theta)\cdot \#Z(G)}\operatorname{vol}_{\overline{\Omega}_\Theta}\left(\mathcal{M}(\Theta)\right).\end{aligned} \tag{9.1.3}$$

By definition of $d_\Theta c$, $\int_\Theta h(c)d_\Theta c = \int_G h(xc_1x^{-1})\,dx$ for any $c_1 \in \Theta$. It follows then by conjugation invariance of Q_t that for any $c \in \Theta$,

$$\begin{aligned}&\int_{G^{2g}\times\Theta} Q_t\left(cb_g^{-1}a_g^{-1}b_ga_g\cdots b_1^{-1}a_1^{-1}b_1a_1\right)\,da_1\cdots db_g d_\Theta c\\ &= \int_{G^{2g}} Q_t\left(cb_g^{-1}a_g^{-1}b_ga_g\cdots b_1^{-1}a_1^{-1}b_1a_1\right)\,da_1\cdots db_g.\end{aligned} \tag{9.1.4}$$

The heat kernel Q_t has a standard expansion in terms of characters of the group G. Using this, it is proven in Lemma 5.5 of [Se5] (see also [Wi1]) that

$$\begin{aligned}&\lim_{t\to 0}\int_{G^{2g}} Q_t\left(cb_g^{-1}a_g^{-1}b_ga_g\cdots b_1^{-1}a_1^{-1}b_1a_1\right)\,da_1\cdots db_g\\ &= \begin{cases}\frac{\pi-\theta_c}{2\sin\theta_c} & \text{if } g=1\\ \sum_{n=1}^{\infty}\frac{\chi_n(c)}{n^{2g-1}} & \text{if } g\geq 2\end{cases},\end{aligned} \tag{9.1.5}$$

where

$$\theta_c \in (0,\pi) \text{ is defined by } \cos\theta_c = \operatorname{Tr}(c)/2, \tag{9.1.6}$$

i.e. c is conjugate to $\begin{pmatrix} e^{i\theta_c} & 0\\ 0 & e^{-i\theta_c}\end{pmatrix}$, and χ_n is the character of the n−dimensional irreducible representation of $SU(2)$:

$$\chi_n(c) = \frac{\sin\left[n\cos^{-1}\left\{\frac{1}{2}\operatorname{Tr}(c)\right\}\right]}{\sin\left[\cos^{-1}\left\{\frac{1}{2}\operatorname{Tr}(c)\right\}\right]} \text{ for every } c \in SU(2)\setminus\{\pm I\}. \tag{9.1.7}$$

Combining this with (9.1.3) we have, with θ_c as in (9.1.6),

$$\operatorname{vol}_{\overline{\Omega}_\Theta}\left(\mathcal{M}(\Theta)\right) = \begin{cases} 2\left|\det(1-\operatorname{Ad}c)^{-1}\right|^{\frac{1}{2}}\operatorname{vol}(\Theta)\frac{\pi-\theta_c}{2\sin\theta_c} & \text{if } g=1\\ 2\left|\det(1-\operatorname{Ad}c)^{-1}\right|^{\frac{1}{2}}\operatorname{vol}(\Theta)\operatorname{vol}(G)^{2g-2}\sum_{n=1}^{\infty}\frac{\chi_n(c)}{n^{2g-1}} & \text{if } g\geq 2\end{cases}. \tag{9.1.8}$$

Next we compute vol(Θ). Fix $c \in \Theta$. If we view $SU(2)$ as a 3-sphere in $\mathbf{R}^4$ in the usual way, the angle between c and I is θ_c. So the surface area of the 2-sphere Θ is

$$\operatorname{vol}(\Theta) = 4\pi\sin^2\theta_c\left[\frac{\operatorname{vol}(SU(2))}{2\pi^2}\right]^{\frac{2}{3}}, \tag{9.1.9}$$

where of course the volume of $SU(2)$ is with respect to the metric $\langle\cdot,\cdot\rangle_{\underline{g}}$ on the Lie algebra of $SU(2)$.

The mapping $e^{i\theta} \mapsto \mathrm{Ad}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}$ is a homomorphism of the circle group onto the group of rotations in a plane in $\underline{g}$ and has kernel $\{\pm 1\}$; therefore $\mathrm{Ad}\, c$ is rotation by angle $2\theta_c$ in a plane orthogonal to the maximal torus of c, and so

$$|\det(1-\mathrm{Ad}\, c)^{-1}|^{\frac{1}{2}} = \det\begin{bmatrix} 1-\cos 2\theta_c & \sin 2\theta_c \\ -\sin 2\theta_c & 1-\cos 2\theta_c \end{bmatrix}^{-\frac{1}{2}} = \frac{1}{2\sin\theta_c} \tag{9.1.10}$$

Substituting this into (9.1.8) yields the volume formula (9.1.1).

(iii) is a special case of Theorem 8.3, bearing in mind that now $\mathcal{M}(\Theta)^0 = \mathcal{M}(\Theta)$, and that, by (ii), this has finite volume. □

9.2 Remarks.

(i) As already noted,the cases $\Theta = \{I\}$ and $\Theta = \{-I\}$ are essentially contained in the study of the moduli space of flat connections over the trivial $SU(2)$–bundle and the non-trivial $SO(3)$–bundle over the closed surface of genus g dealt with in [Se4,5]. The following observations are based on [Se4,5]. The moduli space $\mathcal{M}(I)$ consists of a number of strata whose structure and volumes have been determined in [Se4,5]. The moduli space $\mathcal{M}(-I)$ consists of one point if g = 1; if g $\geq$ 2 then $\mathcal{M}(-I)$ is a smooth connected 3(2g – 2)–dimensional manifold. The 2–form $\overline{\Omega}_\Theta$ is symplectic and the corresponding volume is obtained by considering

$$\lim_{t\to 0+}\int_{G^{2g}} Q_t\left(-I\cdot b_g^{-1}a_g^{-1}b_g a_g\cdots b_1^{-1}a_1^{-1}b_1 a_1\right)\, da_1\cdots db_g;$$

this leads to the value $2\cdot \mathrm{vol}\,(SU(2))^{2g-2}\sum_{n=1}^{\infty}(-1)^{n-1}/n^{2g-2}$ (which is the same as is obtained if we set vol(Θ) and the determinant factor equal to 1 in the formula (9.1.8)).

(ii) If in the first formula in (9.1.1) (the case g = 1) we set $\theta_c = 0$, we obtain the value $2\pi^2\left[\frac{\mathrm{vol}(SU(2))}{2\pi^2}\right]^{\frac{2}{3}}$ which is exactly the volume computed in the closed-surface theory of [Se5] (and in Lemma 3.11 in [Se4]). Such a comparison cannot be made for g $\geq$ 2 since in this case the dimension of $\mathcal{M}(\Theta)$ collapses at $\Theta = \{I\}$.

(iii) If we set g = 1 in the <u>second</u> formula (i.e. the one for g $\geq$ 2) in (9.1.1) and use the trigonometric sum formula $\sum_{n=1}^{\infty}(\sin n\theta)/n = (\pi-\theta)/2$ for $\theta \in (0,\pi)$, then what results is the <u>first</u> formula in (9.1.1) (i.e. the one for g = 1); thus the second formula actually covers all cases.

Appendix

We will quote some results from [Se2,3] and indicate briefly how they are applied in Sects. 3 and 5 to the construction of μ_T^c and μ_T^Θ.

The basic conditional proability result we need is (from [Se2]):

A1. Theorem. *Let $(\Omega_i, \mathcal{F}_i, P_i)$, for $i = 1,2$ be probability spaces, where Ω_1 and Ω_2 are complete separable metric spaces and the corresponding $\mathcal{F}_i$ are the Borel σ–algebras.*

Let G be a compact Lie group. Consider random variables $h_i : \Omega_i \to G$, for $i = 1, 2$, each having positive density with respect to Haar measure on G. Then there is a unique assignment

$$(\mathcal{F}_1 \otimes \mathcal{F}_2) \times G \to [0,1] : (E, z) \mapsto P(E|h_1h_2 = z)$$

such that the following hold:

(i) $E \mapsto P(E|h_1h_2 = z)$ *is a probability measure for every* $z \in G$.

(ii) $z \mapsto P(E|h_1h_2 = z)$ *is measurable for every* $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, *and*

$$\int_V P(E|h_1h_2 = z)\, dp_{12}(z) = (P_1 \otimes P_2)(E \cap \{h_1h_2 \in V\})$$

holds for every Borel $V \subset G$, *where* p_{12} *is the probability measure on* G *describing the distribution of* h_1h_2

(iii) $z \mapsto P(A \times B|h_1h_2 = z)$ *is continuous for every* $A \in \mathcal{F}_1$ *and* $B \in \mathcal{F}_2$. *Moreover,*

$$P(\{h_1h_2 = z\}|h_1h_2 = z) = 1$$

Using the conditional measure described above, we can construct another conditional measure, which is the one we use for the measures μ_T^c and μ_T^Θ.

A2. Definition. *Let $(\Omega_i, \mathcal{F}_i, P_i)$, for $i = 1, 2, 3$, be probability spaces, with $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ being as in Theorem A1. Let G be a compact Lie group and consider $G-$valued random variables h_i on Ω_i, with h_1 and h_2 having positive densities with respect to Haar measure on G. Consider the product space $(\Omega, \mathcal{F}) = (\Omega_1, \mathcal{F}_1) \times (\Omega_2, \mathcal{F}_2) \times (\Omega_3, \mathcal{F}_3)$; for $E \in \mathcal{F}$ we define*

$$P(E|h_1h_2 = h_3) \stackrel{\text{def}}{=} \frac{1}{Z} \int_{\Omega_3} dP_3(\omega_3)\, P\left(E^{\omega_3}|h_1h_2 = h_3(\omega_3)\right) \rho_{12}\left(h_3(\omega_3)\right) \qquad (A2.1)$$

where ρ_{12} is the density of h_1h_2 with respect to Haar measure on G, and $E^{\omega_3} = \{(\omega_1, \omega_2) : (\omega_1, \omega_2, \omega_3) \in E\}$, and

$$Z = \int_{\Omega_3} \rho_{12}\left(h_3(\omega_3)\right) dP_3(\omega_3), \qquad (A2.2)$$

Then $P(\cdot|h_1h_2 = h_3)$ makes sense, is a probability measure on $(\Omega, \mathcal{F})$, and satisfies

$$P(\{h_1h_2 = h_3\}|h_1h_2 = h_3) = 1. \qquad (A2.3)$$

Thus $P(\cdot|h_1h_2 = h_3)$ really lives on the subset of Ω where $h_1h_2 = h_3$.

A3. Construction of μ_T^c. Fix a positive real number T. We shall describe the details of the construction of μ_T^c, for any $c \in G$, in terms of the conditional probability measure described in Definition A2. For the sake of notational simplicity we shall only describe the case considered in Sect. 3, i.e. of the torus with one hole. The general case is exactly analogous and is discussed briefly in Sect. A5 below.

For the construction of μ_T^c, divide the disk D into an upper half D_U and a lower half D_L. The boundaries ∂D_L and ∂D_U, when considered as loops, will be taken to start from the center O of the disk D. Thus, with L_0 being the radial path from O to $x_0 = (1, 0) \in \partial D$, we have

$$h(\partial D_L; \omega)h(\partial D_U; \omega) = h(\overline{L}_0 \cdot \partial D \cdot L_0; \omega) \qquad (A3.1)$$

for any connection ω over D.

Equip D with the measure which corresponds, via the quotient map $q : D \to \Sigma$, to the Riemannian area measure on Σ. For any subset $A \subset D$ (such as $A = D_U$ or $A = D_L$) let $L^2(A; \underline{g})$ be the corresponding Hilbert space of $\underline{g}$-valued square-integrable functions on D vanishing outside A.

Let Ω_1 be a Hilbert-Schmidt closure of $L^2(D_U; \underline{g})$ and P_1^T, or simply P_1, the standard Gaussian measure, with variance scaled by T, on the Borel σ-algebra $\mathcal{F}_1$ of Ω_1:

thus for each $f \in L^2(D_U; \underline{g})$ there is a Gaussian random variable $\tilde{f}$ on Ω_1, such that for any $f_1, f_2 \in L^2(D_U; \underline{g})$, $\langle \tilde{f}_1, \tilde{f}_2 \rangle_{L^2(P_1)} = T\langle f_1, f_2 \rangle_{L^2(D_U;\underline{g})}$.

The $\underline{g}$-valued white-noise referred to in Sect. 3.2 (or at least the part of it over D_U) is obtained, for instance, by choosing any orthonormal basis $e_1, \ldots, e_d$ of $\underline{g}$ and setting $F^\omega(E) = \sum_{i=1}^d (1_E e_i)\tilde{} e_i$ for every Borel $E \subset D_U$.

Let $(\Omega_2, \mathcal{F}_2, P_2)$ be the corresponding space of D_L (we are suppressing the superscript T in P_2^T).

We modify the definition of Ω_{disk} given in Sect. 3.2 to:

$$\Omega_{\text{disk}} = \Omega_U \times \Omega_L. \tag{A3.2}$$

Finally, let $\Omega_3 = G^2$, and let P_3 be the unit-mass Haar measure on G^2. The functions h_i are as follows:

$$h_1(\omega_1) = h(\partial D_U; \omega_1), \tag{A3.3}$$

$$h_2(\omega_2) = h(\partial D_L; \omega_2), \tag{A3.4}$$

$$h_3(a, b) = cb^{-1}a^{-1}ba, \tag{A3.5}$$

In $(A3.4)$ and $(A3.5)$, the left sides are defined in terms of solutions of the stochastic differential equation (3.2.1) and, as pointed out after (3.2.1), are G-valued random variables with densities $Q_{T|D_U|}(\cdot)$ and $Q_{T|D_L|}(\cdot)$, respectively.

As in Sect. 3.3, we define

$$\Omega_c = \Omega_{\text{disk}} \times G^2, \tag{A3.6}$$

and we define the measure μ_T^c by

$$\mu_T^c = P(\cdot | h_1 h_2 = h_3), \tag{A3.7}$$

where the right side is the probability measure specified above in Definition A2.

Although Ω_c does not depend on c, it follows from $(A2.3)$ that the measure μ_c really lives on the subspace of ω for which the constraint

$$h(\overline{L}_0 \cdot \partial D \cdot L_0; \omega) = cb^{-1}a^{-1}ba$$

holds (the left side is defined to be the product $h(\partial D_L; \omega_2)h(\partial D_U; \omega_1)$, following $(A3.1)$).

From the expression in $(A2.2)$ for the 'normalizer' $Z = Z_T(c)$, and from the observations made above concering the densities of h_1 and h_2, we have:

$$Z_T(c) = \int_{G^2} Q_{T|\Sigma|}(cb^{-1}a^{-1}ba)\, da\, db,$$

which is the same as our earlier value (3.4.2).

A4. *Construction of* μ_T^Θ. The construction of μ_T^Θ, where Θ is any conjugacy class in G, is similar to that for μ_T^c. The only difference is that now we set

$$\Omega_3 = G^2 \times \Theta. \tag{A4.1}$$

take the measure P_3 to be (unit-mass Haar on G^2)$\times$(the $G-$invariant unit-mass measure on Θ), take F_3 to be given by $h_3(a,b,c) = cb^{-1}a^{-1}ba$, and define

$$\Omega_\Theta = \Omega_{\text{disk}} \times G^2 \times \Theta. \tag{A4.2}$$

The spaces $(\Omega_i, \mathcal{F}_i, P_i)$, for $i = 1,2$, and the corresponding variables h_1 and h_2 are as for μ_T^c. With this, we define

$$\mu_T^\Theta \stackrel{\text{def}}{=} P(\cdot|h_1h_2 = h_3). \tag{A4.3}$$

The corresponding normalizer $Z = N_T(\Theta)$ is then:

$$N_T(\Theta) = \int_{G^2\times\Theta} Q_{T|\Sigma|}(cb^{-1}a^{-1}ba)\, da\, db\, d_\Theta c \tag{A4.4}$$

which agrees with the value in (3.4.5).

A5. *Other surfaces.* For general compact surfaces (as described in Sect. 5), Ω_3 is taken to be $G^{\sigma g} \times G^p$ (with notation as in Sect. 5; σ is 1 if Σ is unorientable, 2 otherwise), and h_3 is taken to be the appropriate function of the a_i, b_i, c_i given by the right side of (5.3.6).

Finally, we quote from (Proposition 4.5 of) [Se2] the exact expectation-value formula which was alluded to in Sect. 3.4.

A6. Proposition. *With notation and hypotheses as in Sects. A2 and A3, let* $\phi = (\phi_1, \dots, \phi_m) : \Omega_1 \to G^m$ *and* $\psi = (\psi_1, \dots, \psi_n) : \Omega_2 \to G^n$ *be measurable functions. Suppose that* $\phi_k \cdots \phi_1 = h_1$ *and* $\psi_l \cdots \psi_1 = h_2$, *where the* h_i *are as in Sect. A1. Suppose also that* ϕ *has a bounded density* ρ_ϕ *on* G^m, *and* ψ *has a bounded density* ρ_ψ *on* G^n. *Then for any bounded measurable function* f *on* $G^m \times G^n \times \Omega_3$, *we have* (with $\omega = (\omega_1, \omega_2, \omega_3)$):

$$\begin{aligned}
&\int f\Big(\phi(\omega_1), \psi(\omega_2), \omega_3\Big)\, dP(\omega|h_1h_2 = h_3)\\
&= \frac{1}{Z}\int f(x,y,\omega_3)\rho_\phi(x)\rho_\psi(y)\delta\Big(x_k \cdots x_1 y_l \cdots y_l h_3(\omega_3)^{-1}\Big)\, dP_3(\omega_3),
\end{aligned}$$

where $\delta(\cdot)$ *means that we can drop any* x_j *(with* $1 \le j \le k$*) or* y_j *(with* $1 \le j \le l$*) from the integration and replace it in the integrand with the value which makes* $x_k \cdots x_1 y_l \cdots y_1 = h_3(\omega_3)$, *and* Z *is the normalizing constant given in* (A2.2).

In applications, we work typically with random variables of the form $h(\kappa;\omega)$, where κ is an admissible loop in D based at O; breaking κ into pieces in D_L and pieces in D_U (and adjoining appropriate additional radial segments) we can express $h(\kappa;\omega)$ as a product of variables $h(\kappa';\omega)$, where κ' is a loop in D_L or in D_U. Thus Proposition A6 is applicable in situations, where ϕ_i and ψ_i are holonomy variables $h(\kappa';\cdot)$.

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