Tôhoku Math. Journ.
Vol. 19, No. 3, 1967

WEIGHTED SUMS OF CERTAIN DEPENDENT RANDOM VARIABLES

KAZUOKI AZUMA

(Received June 15, 1967)

1. Let $(\Omega, \mathfrak{A}, \mathrm{P})$ be a probability space and $(\mathfrak{A}_n)_{n=1,2,\dots}$ be an increasing family of sub σ-fields of $\mathfrak{A}$ (we put $\mathfrak{A}_0 = (\phi, \Omega)$). Let $(x_n)_{n=1,2,\dots}$ be a sequence of bounded martingale differences on $(\Omega, \mathfrak{A}, \mathrm{P})$, that is, $x_n(\omega)$ is bounded almost surely (a.s.) and $\mathrm{E}\{x_n | \mathfrak{A}_{n-1}\} = 0$ a.s. for $n = 1, 2, \cdots$. It is easily seen that this sequence has the following properties [G] and [M], which have been introduced by Y. S. Chow ([1]) in an analogous form and by G. Alexits ([4]), respectively, and may be of independent interest.

[G] (x_n) is a sequence of martingale differences and there exist non negative constants c_n such that for every real number t

$$\mathrm{E}\{\exp(tx_n) | \mathfrak{A}_{n-1}\} \leqq \exp(c_n^2 t^2/2) \text{ a.s. } (n = 1, 2, \cdots).$$

For each n, the minimum of those c_n is denoted by $\tau(x_n)$.

[M] $$|x_n(\omega)| \leqq K_n \quad \text{a.s. for } n = 1, 2, \cdots$$

and $\mathrm{E}\{x_{i_1} x_{i_2} \cdots x_{i_k}\} = 0$ for $i_1 < i_2 < \cdots < i_k\,; k = 1, 2, \cdots$.

In this note we investigate the asymptotic behavior of the weighted sums of those random variables. In §3 we will deal with the class [M] and in §4 with the class [G] and the uniformly bounded case of martingale differences.

2. Preliminary Lemmas.

LEMMA 1. *If (x_n) is a sequence of random variables for which* [M] *holds with $K_n = 1$ for all n, then for every real number t*

$$\mathrm{E}\left\{\exp\left(t \sum_{k=1}^{n} b_{nk} x_k\right)\right\} \leqq \exp\left(\frac{t^2}{2} \sum_{k=1}^{n} b_{nk}^2\right), \tag{2. 1}$$

where $(b_{nk})_{k=1,2,\cdots n;\, n=1,2,\cdots}$ is an arbitrary sequence of real numbers.

PROOF. We may assume that $|b_{nk}| \neq 0$ for $k = 1, 2, \cdots$. Since $|b_{nk} x_k| \leqq |b_{nk}|$ a.s. and the exponential function $\exp(t b_{nk} x_k)$ is convex, we have