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# NUMERICAL CALCULATION OF THE GENERALIZED SINE AND COSINE INTEGRAL

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ABSTRACT. A method is presented for the efficient calculation of the generalized sine and cosine integral. The evaluation is done by Čebyšev series with two nested recursions.

**1. Introduction.** The generalized sine and cosine integrals (see Table 1) are defined as

$$\mathrm{Si}\,(x,\alpha) := \int_0^x \frac{\sin t}{t^\alpha}\,dt, \quad 0 \le x, 0 < \alpha < 2, \tag{1}$$

$$\mathrm{Ci}\,(x,\alpha) := \int_0^x \frac{\cos t}{t^\alpha}\,dt, \quad 0 \le x, 0 < \alpha < 1. \tag{2}$$

Both functions were extensively studied in all details with respect to their analytical behaviour in Kreyszig [**5**]. In an earlier paper by Walther [**7**] the generalized sine integral was already used to study the Gibbs's phenomenon of Fourier series.

Special cases of both integrals are well known.

For $\alpha = 1$ we obtain the "ordinary" sine integral $\mathrm{Si}\,(x) := \int_0^x (\sin t/t)\,dt$ and for $\alpha = (1/2)$ the Fresnel integrals

$$\int_0^x \frac{\sin t}{\sqrt{t}}\,dt = 2\int_0^{\sqrt{x}} \sin\tau^2\,d\tau, \qquad \int_0^x \frac{\cos t}{\sqrt{t}}\,dt = 2\int_0^{\sqrt{x}} \cos\tau^2\,d\tau.$$

A close relationship exists between both integrals to the hypergeometric function ${}_1\mathrm{F}_2$, the incomplete Gamma function $\gamma$ and the confluent

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